Wierzba, May 2004

Physics of the cracking whip

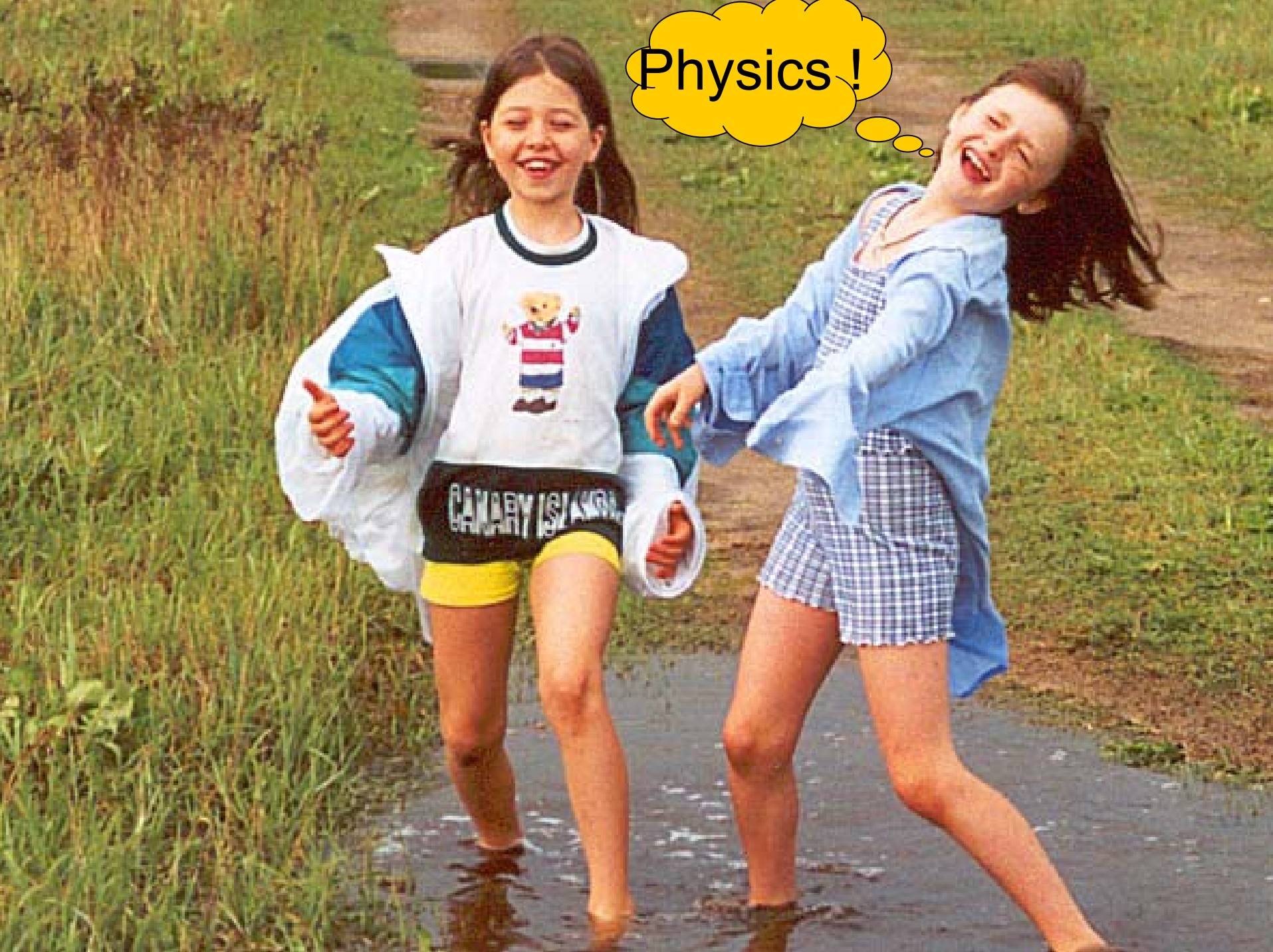
Piotr Pierański and Waldemar Tomaszewski

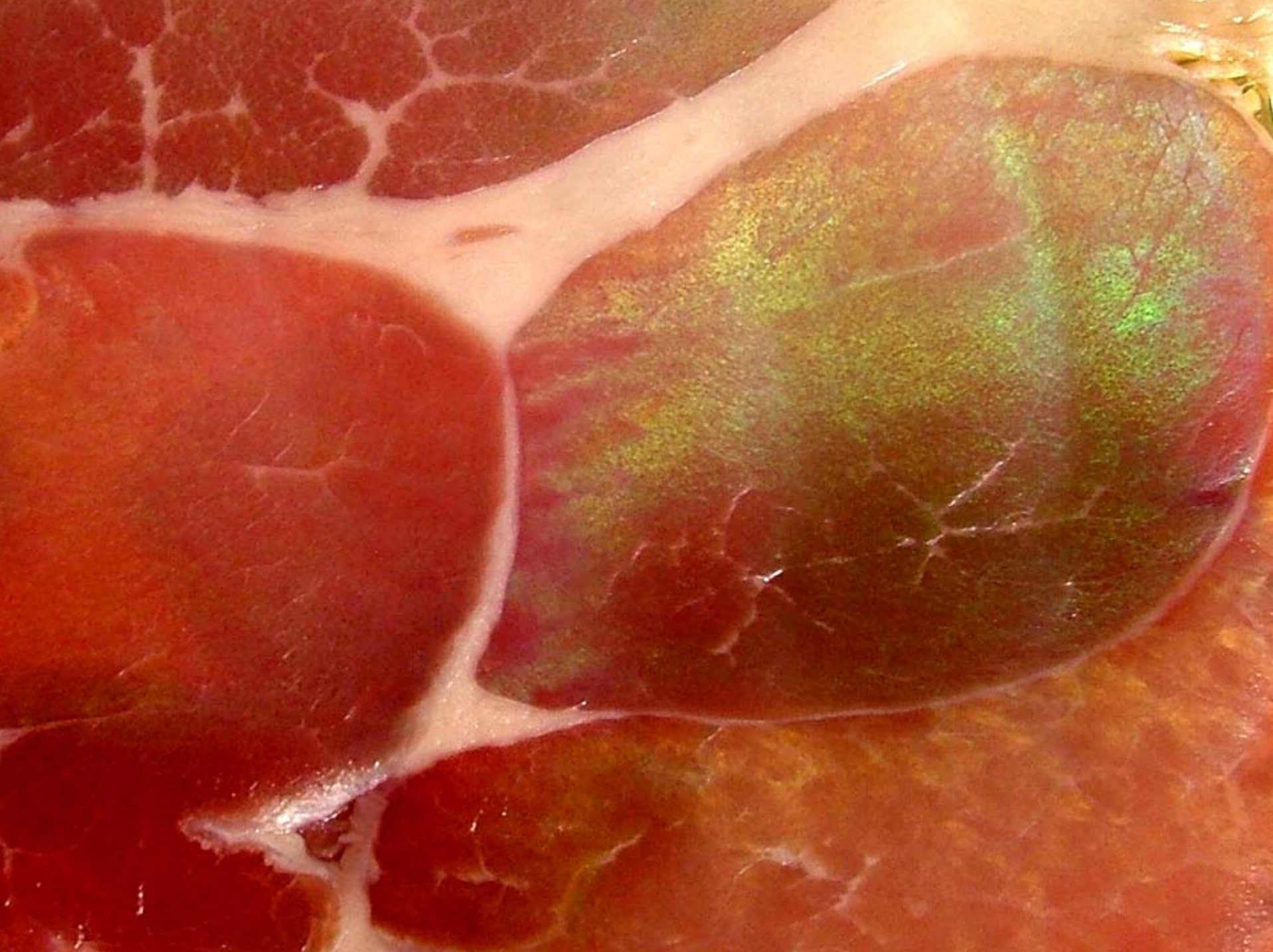
Laboratory of Computational and Semiconductor Physics

Poznan University of Technology

Physics

around us


Physics !




$P=\sigma T^4$
$\lambda_{max}=c/T$


tendril


Tendril perversion
LEFT
RIGHT
Lk(K,U) = Tw(K,U) + Wr(K) = const.

Shape of a Cracking Whip

Alain Goriely[2,1,*] and Tyler McMillen[2,†]

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[2]*Program in Applied Mathematics, University of Arizona, Tucson, Arizona 85721*

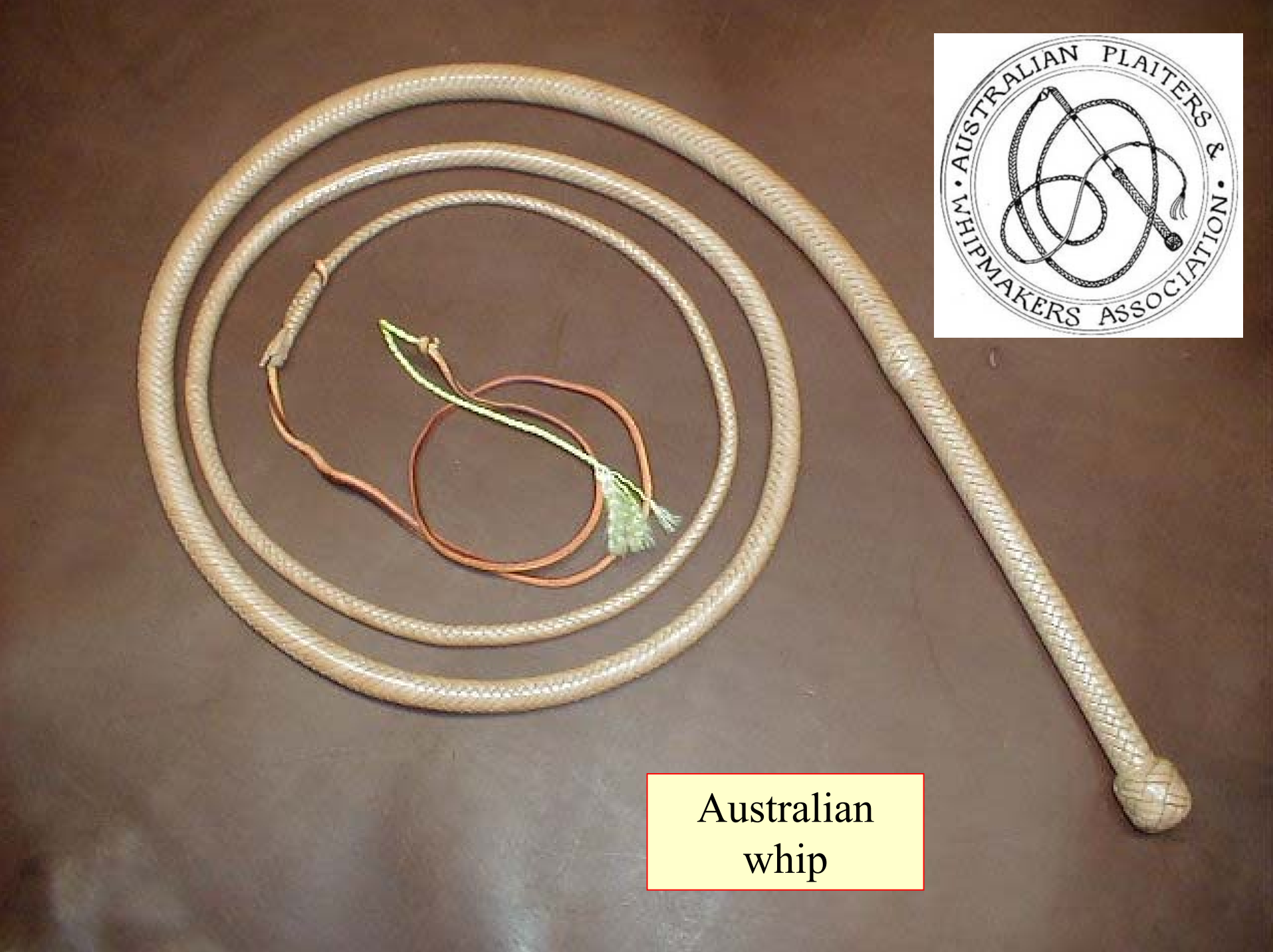
(Received 4 March 2002; published 3 June 2002)

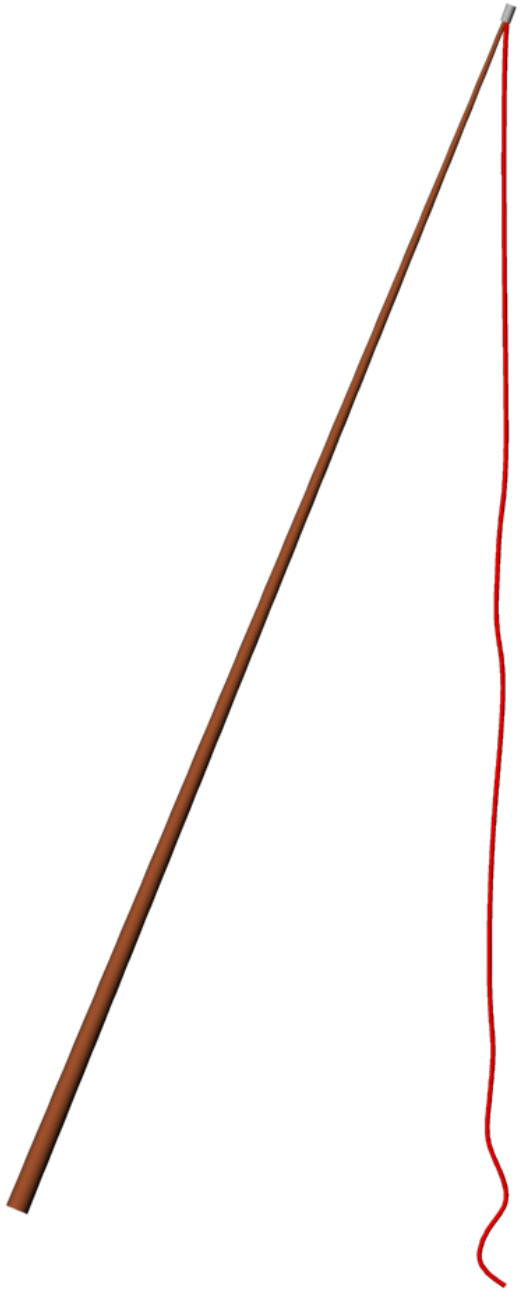
The crack of a whip is produced by a shock wave created by the supersonic motion of the tip of the whip in the air. A simple dynamical model for the propagation and acceleration of waves in the motion of whips is presented. The respective contributions of tension, tapering, and boundary conditions in the acceleration of an initial impulse are studied theoretically and numerically.

DOI: 10.1103/PhysRevLett.88.244301 PACS numbers: 46.70.Hg, 05.45.–a, 46.40.–f

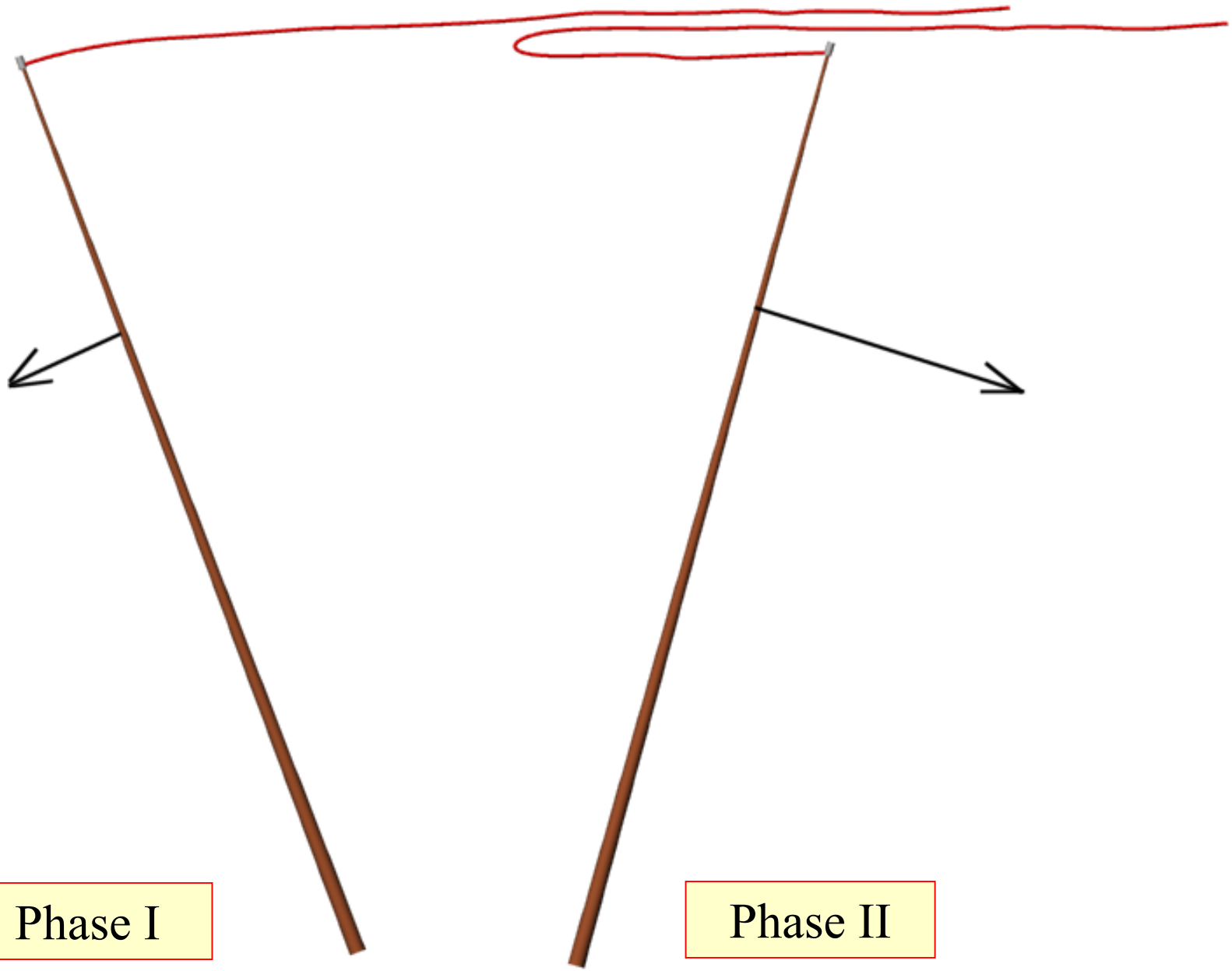
Whips are among the most misunderstood and misrepresented objects in today's culture. Whips are usually thought of either as weapons (like the ones used by Zorro and Indiana Jones) or as instruments of torture associated with slavery and perverse activities. The world of whips is divided into two main categories, the "pain-making" whips which are short, bulky, and often multithreaded (such as the infamous cat-o'-nine-tails) and the "noise-making" whips which are long, tapered, and single threaded (such as the stockwhip or the bullwhip). It is the latter category which

The cracking whip phenomenon


AUSTRALIAN PLAITERS & WHIPMAKERS ASSOCIATION
Australian whip



Polish
whip


Phase I
Phase II

History

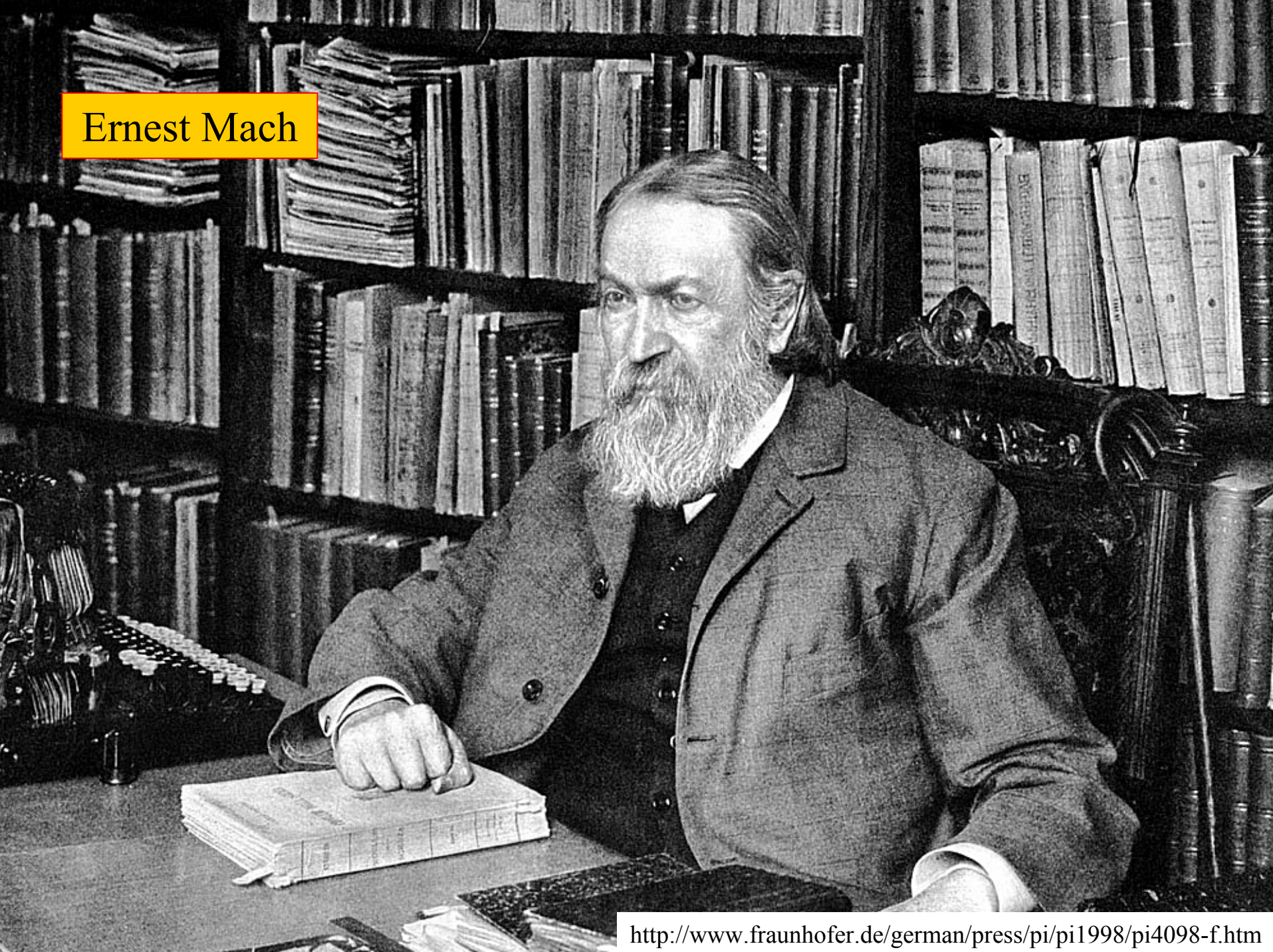
Otto Lummer (1860-1925)

Professor at the Breslau University

1905 - first speculations on the origin of the cracking sound.

See:

*Über die Theorie des Knalls***.** Schlesische Gesellschaft für vaterländische Kultur, **83**, II:2 (1905)



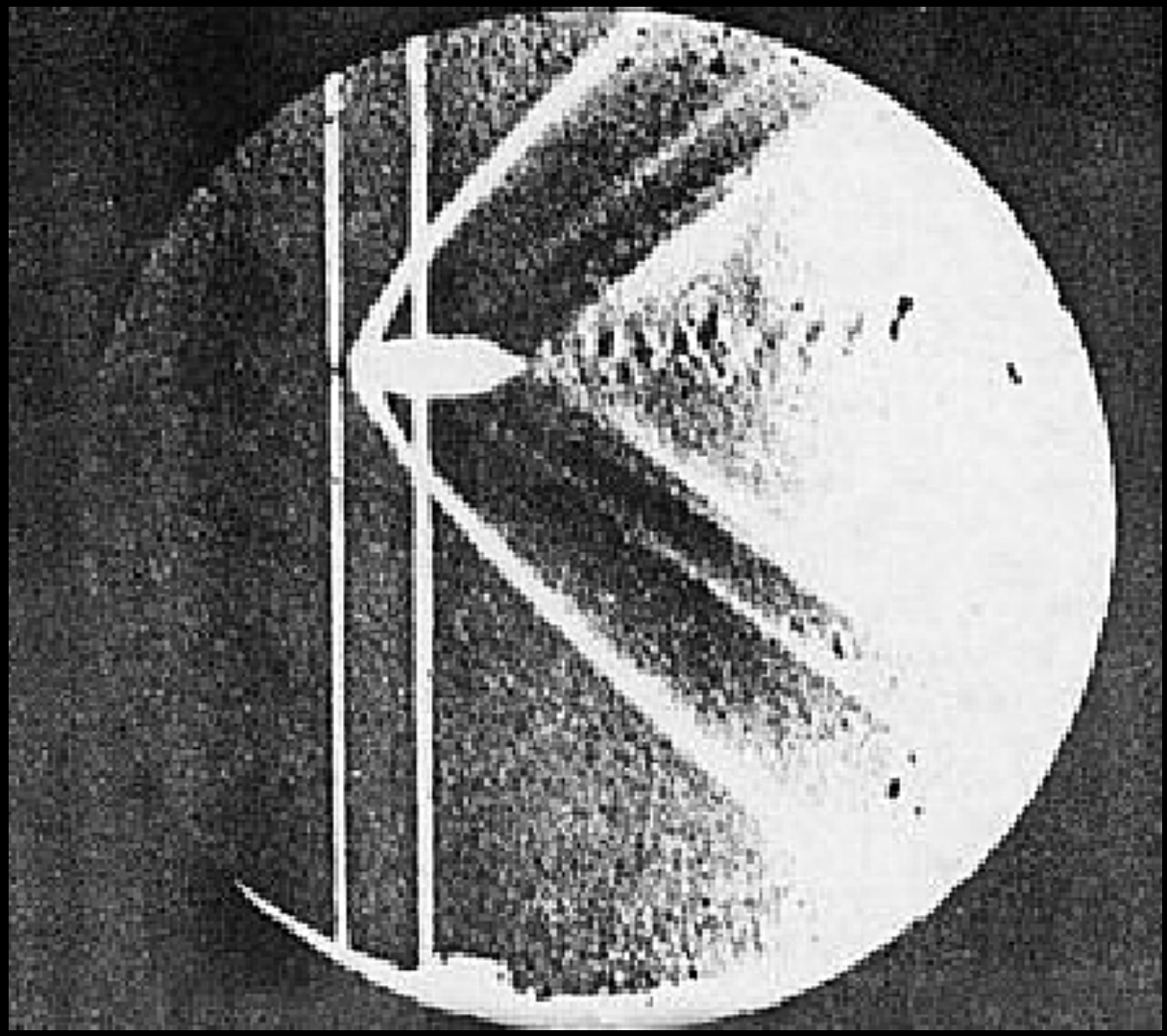
http://www.fraunhofer.de/german/press/pi/pi1998/pi4098-f.htm

Born: 18.02.1838, Chirlitz-Turas, Moravia
Died: 19.02.1916, Haar

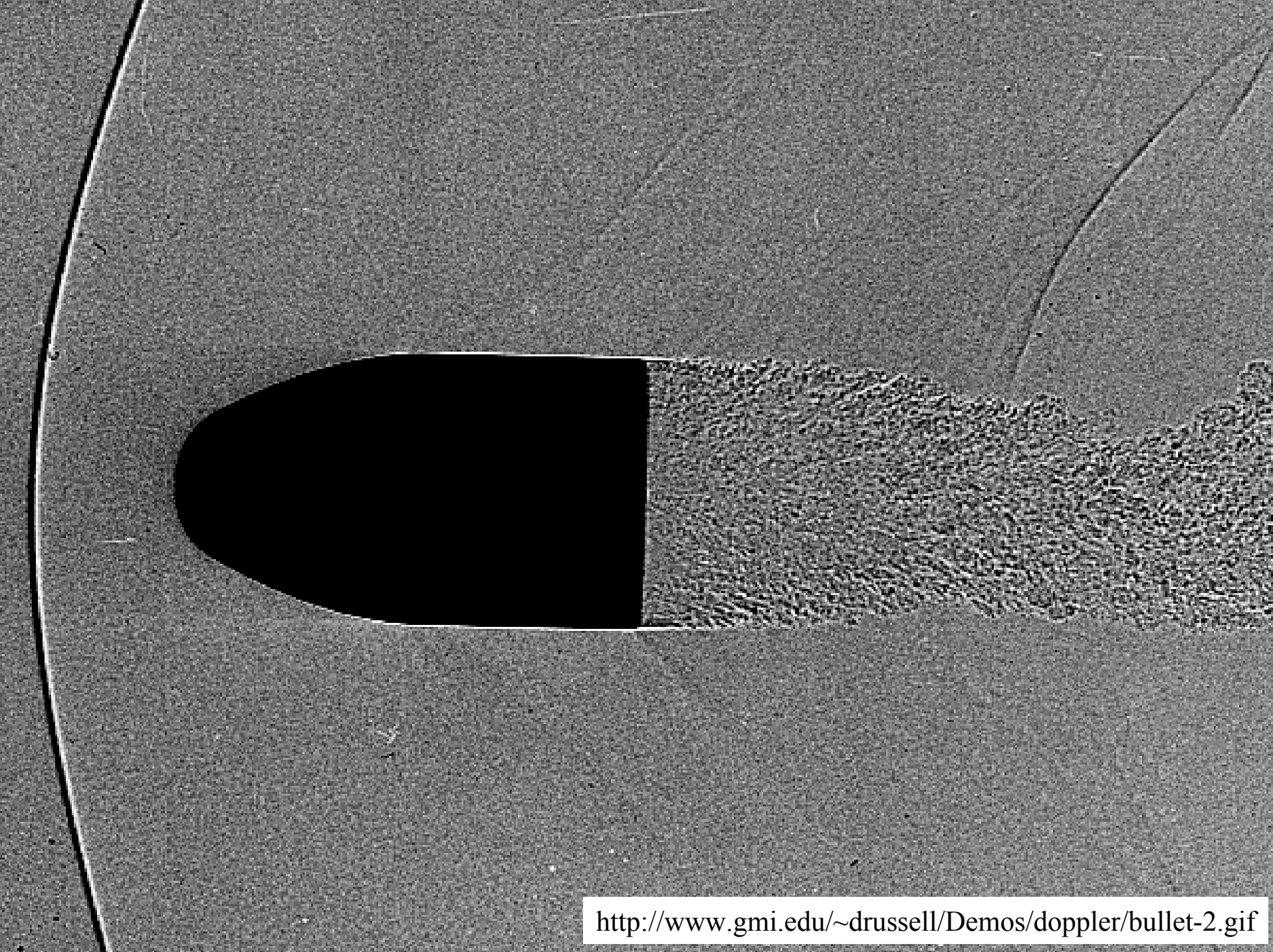
1855 - Wien University
1860 - PhD in physics (22 years old)
1864 - moving to Graz
1867 - moving to Prage; chair at the Karol Universit
1873 - 1893 development of the Schlieren Photographie
1887 - definition of the „Mach number”
1895 - return to Wien (after 28 years !)
1901 - pension
1905 - „*Erkenntnis und Irrtum*”
1910 - autobiography

„A statement is sensible if it can be verified experimentally.”

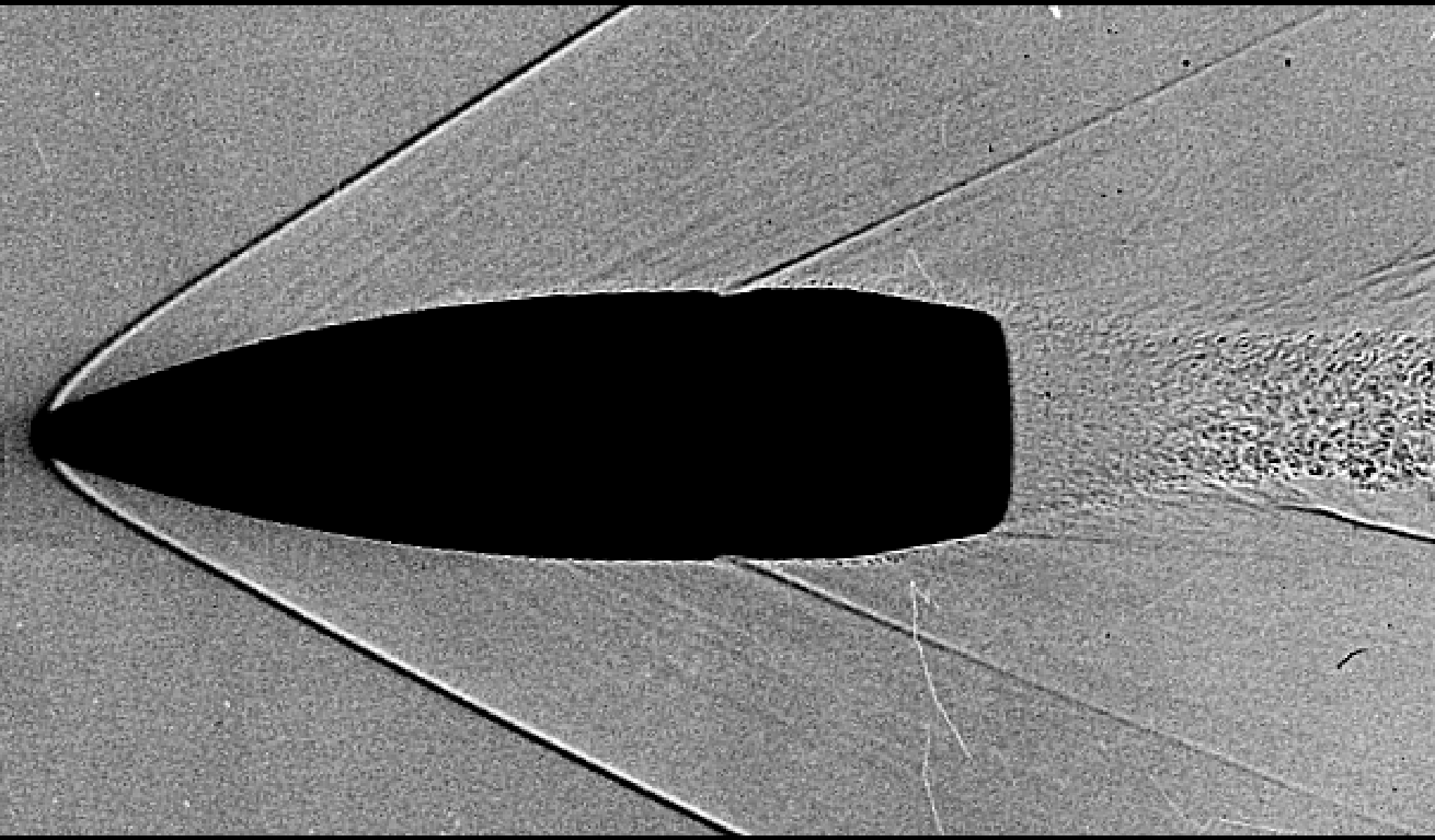
http://history.nasa.gov/SP-4219/Chapter3.html



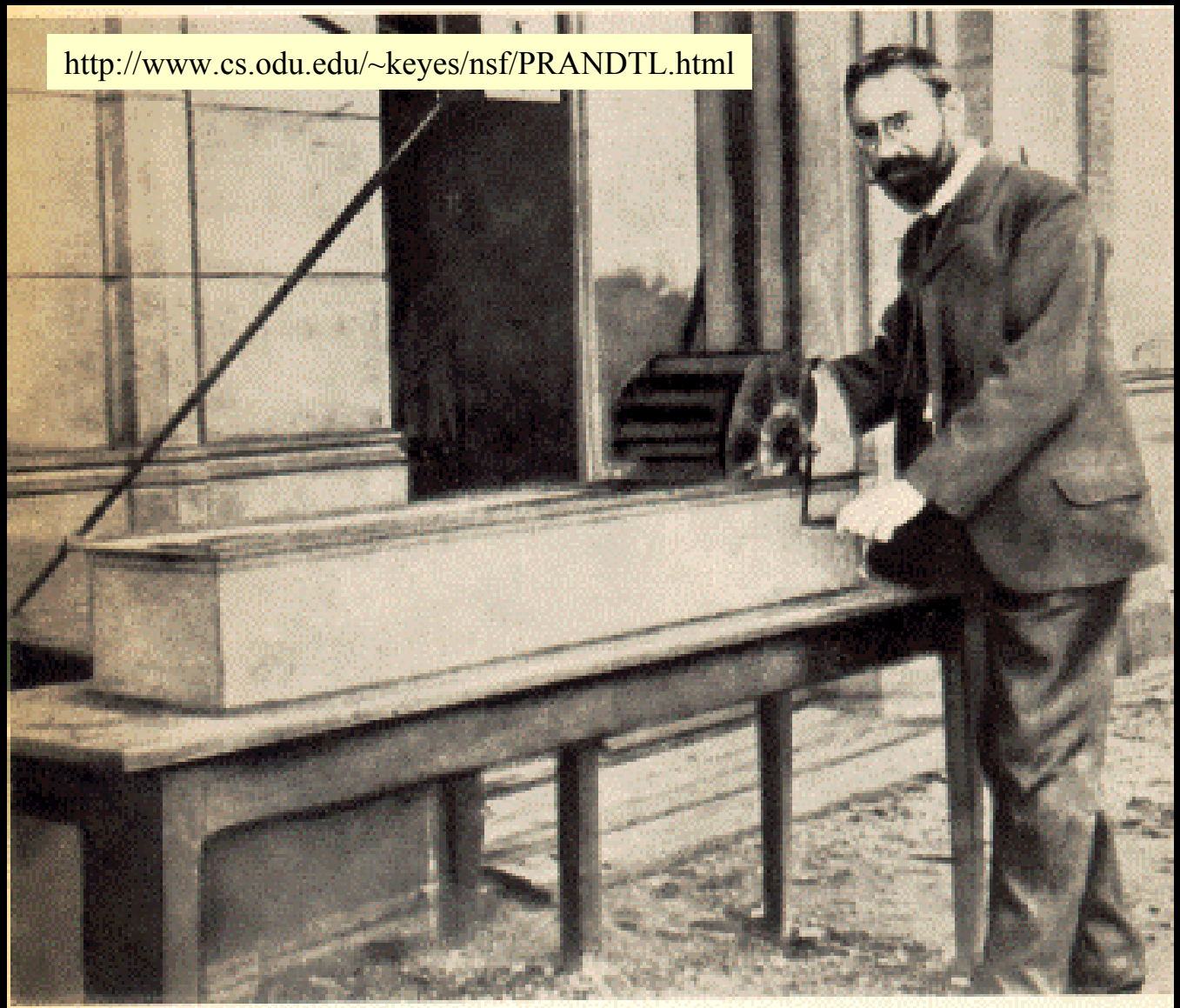
Ernst Mach and Peter Salcher 1887 r.



http://www.gmi.edu/~drussell/Demos/doppler/bullet-2.gif



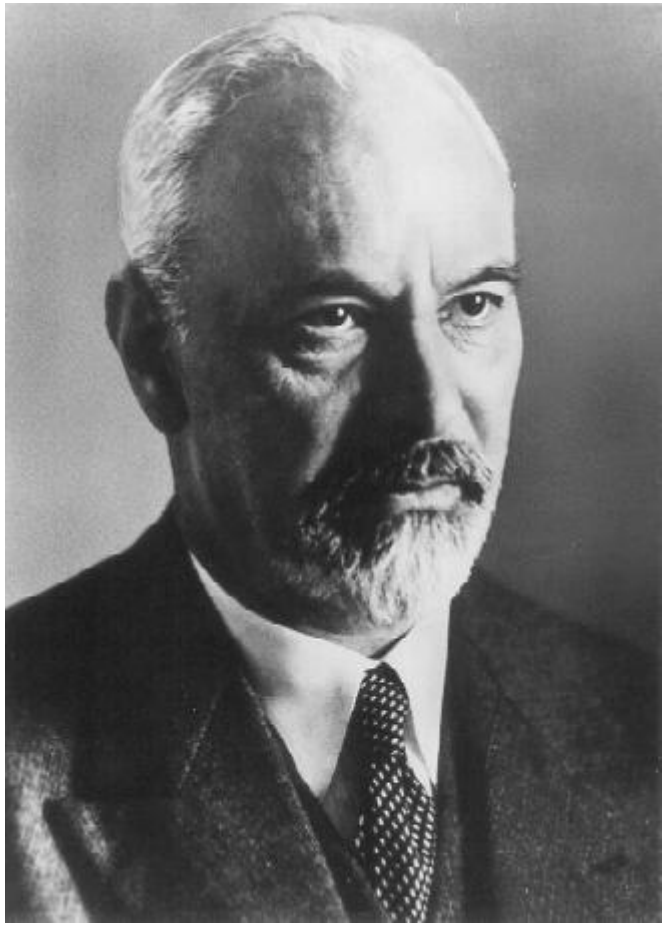
http://www.kettering.edu/~drussell/Demos/doppler/bullet-3.gif



http://www.cs.odu.edu/~keyes/nsf/PRANDTL.html

LUDWIG PRANDTL 1904

an seinem handbetriebenen Wasserkanal in Hannover



LUDWIG PRANDTL 1875-1953

1904 - chair in the applied mechanics in Götingen

1925 - head of the Fluid Mechanics Institute

1904 -discovery of the border layer

1913 -contribution on the Lummer hypothesis

$$Pr = \frac{\nu}{\alpha}$$

Prandtl number

$$\mathrm{Pr} = \frac{\nu}{\alpha}$$

ν - kinematic viscosity
α - thermal diffusivity

$$\nu = \frac{\eta}{\rho}$$

η - dynamic viscosity
ρ - density

$$\alpha = \frac{k}{\rho c_p}$$

k - thermal conductivity
c_p - heat capacity at constant p

First experiment

Journal de Physique, **6**, 366-384 (1927)

LE CLAQUEMENT DU FOUET.

par M. Z. CARRIÈRE
Institut Catholique de Toulouse.

Sommaire. — Un fouet de laboratoire est décrit et son montage détaillé de manière à obtenir, par chronophotographie, les formes instantanées de la corde et de l'onde sonore qu'elle engendre.

Les formes succesives résultent d'une onde transversale se propageant vers l'extrémité libre du fouet et, à cette extrémité, se réfléchissant avec changement de signe. La vitesse de propagation de l'onde et la courbure qui définit l'onde croissent d'abord jusqu'à des valeurs très grandes réalisées à l'instant où s'opère la réflexion (vitesse supérieure à 350 mètres par seconde, courbure d'un millier de dioptries). Ensuite, la vitesse diminue et le rayon de courbure croit à nouveau en valeur absolue (son signe est changé). Au maximum de vitesse et au minimum absolu du rayon de courbure, le bout libre du fouet subit d'énormes variations de direction, de tension, de torsion. Il en résulte, au voisinage, une onde sonore sphérique dont les clichés reproduits montrent la trace. On y retrouve toutes les particularités de l'onde de sillage fournie par les projectiles.

Application de la théorie est faite au fouet de charretier, pour deux façons usitées d'en obtenir le claquement.

Deux claquements très rapprochés dans le temps (fouet à deux ficelles d'inégale longueur) donnent à l'oreille la sensation d'un son de hauteur déterminée que ne donne pas un claquement isolé.

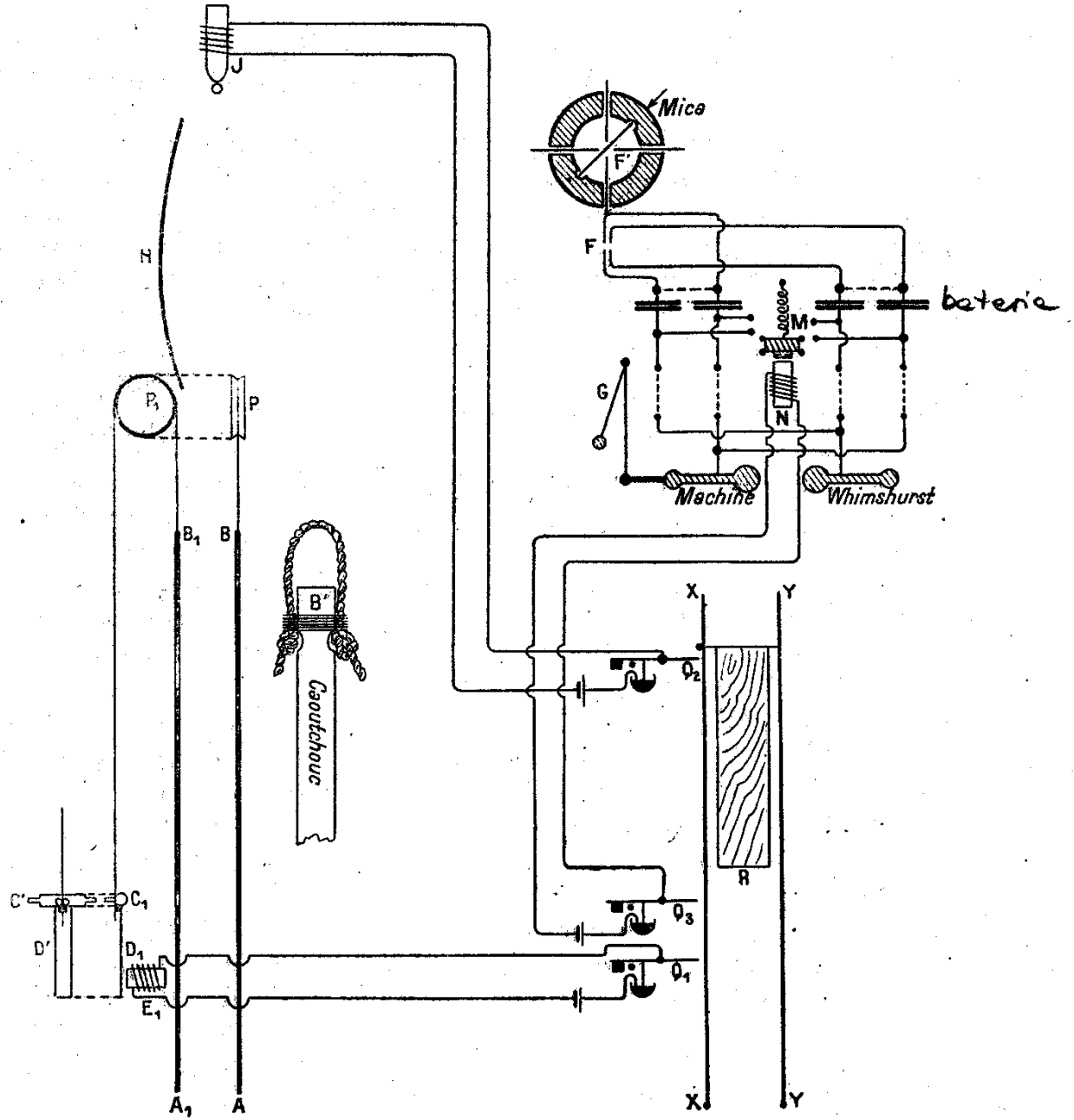

J
Mica
F'
F
H
baterie
M
G
N
P1
P
Machine
Whimshurst
B1
B
B'
Caoutchouc
X
Y
D2
R
D3
D1
C'
C1
D'
E1
A1
A
X
Y

Fig. 1.

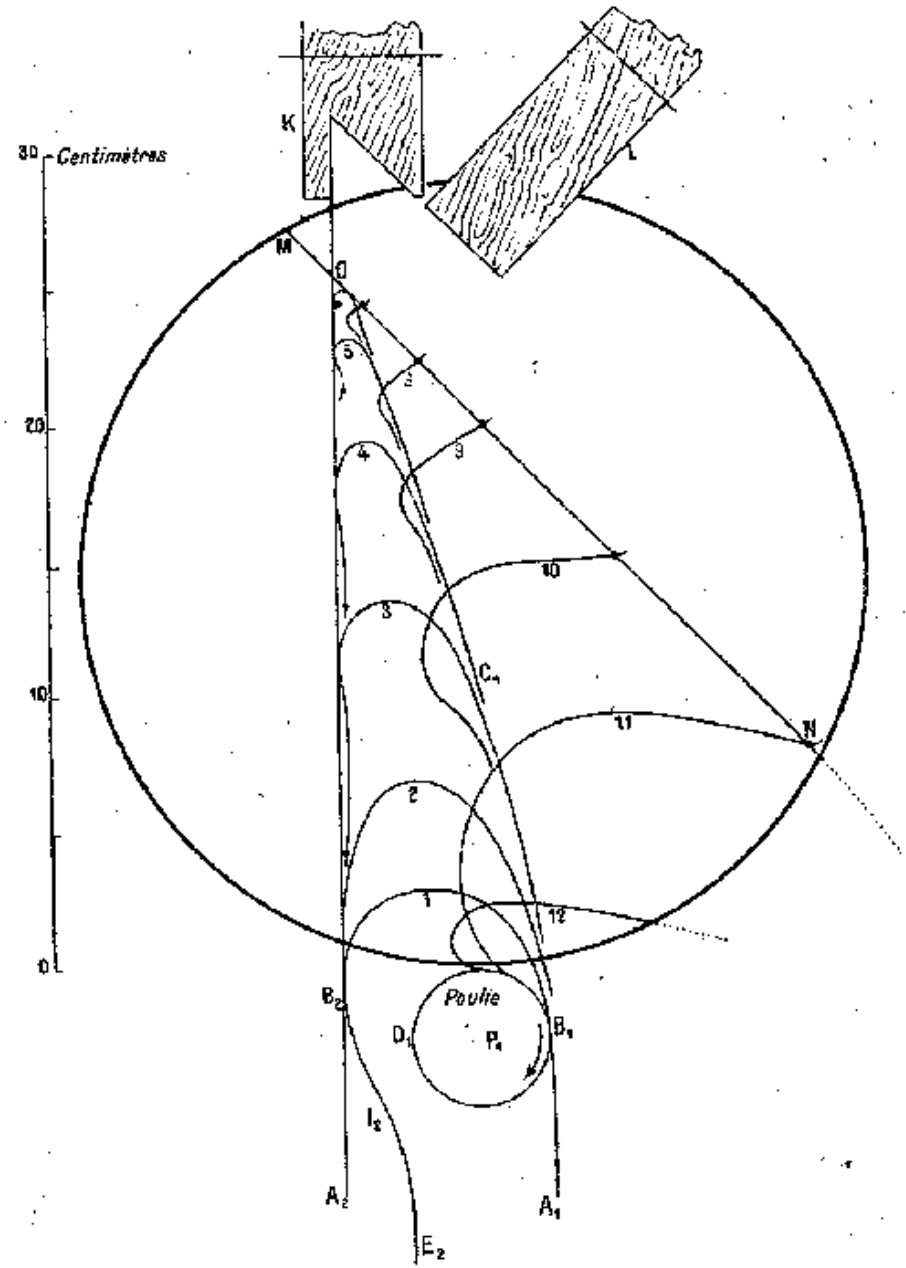

30
Centimètres
20
10
0
K
L
M
O
5
4
3
2
1
10
11
12
N
C₁
Poulie
D₁
P₁
B₂
B₁
I₂
A₂
A₁
E₂

Fig. 2.

Contemporary experiment

Shock Waves (1998) 8: 1–9

Shock Waves

The puzzle of whip cracking – uncovered by a correlation of whip-tip kinematics with shock wave emission*

P. Krehl[1], S. Engemann[1], D. Schwenkel[2]

[1] Ernst-Mach-Institut, Institut für Kurzzeitdynamik der Fraunhofer-Gesellschaft, D-79104 Freiburg, Germany
[2] VKT Video Kommunikation GmbH, D-72787 Pfullingen, Germany

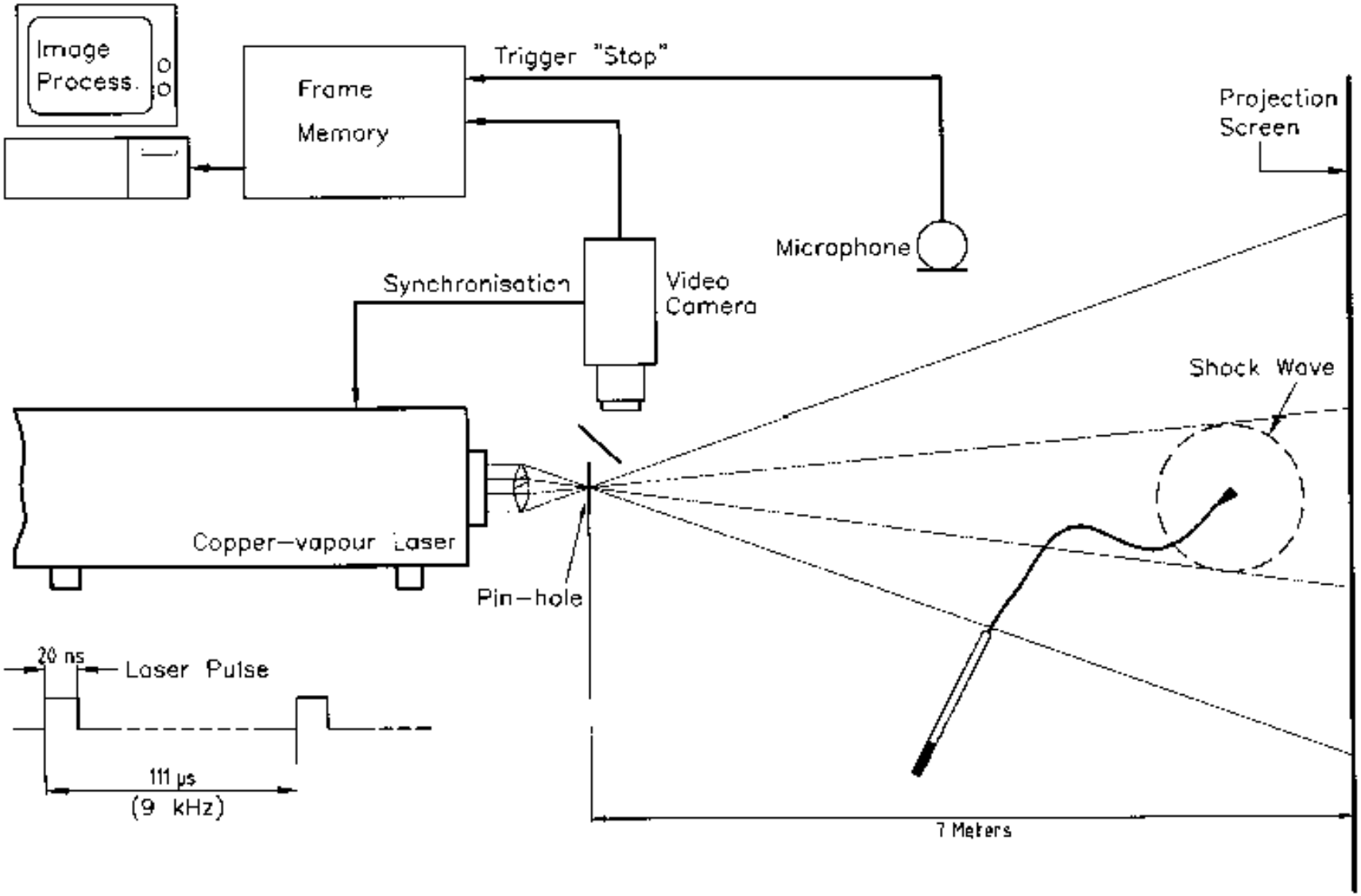
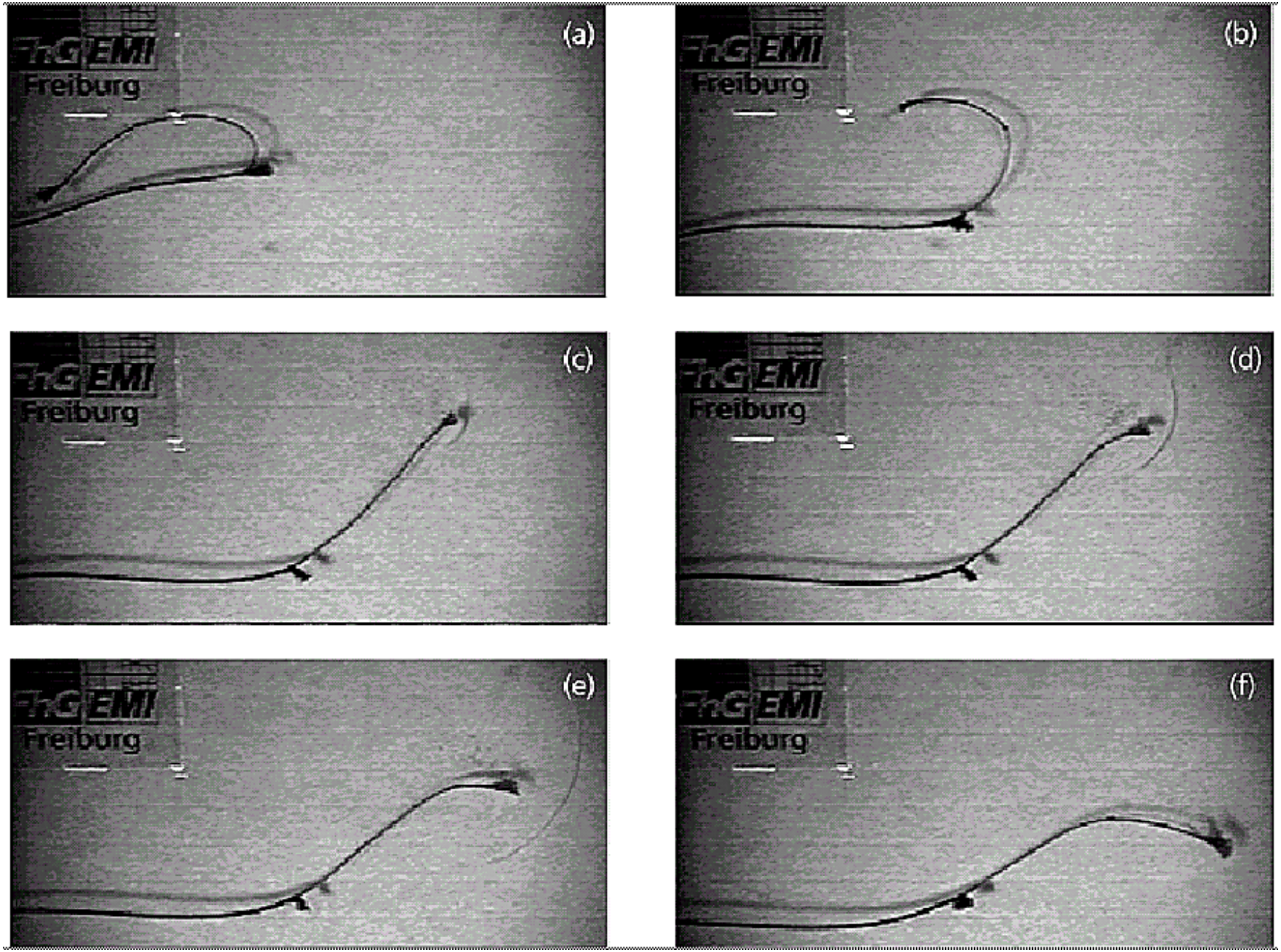
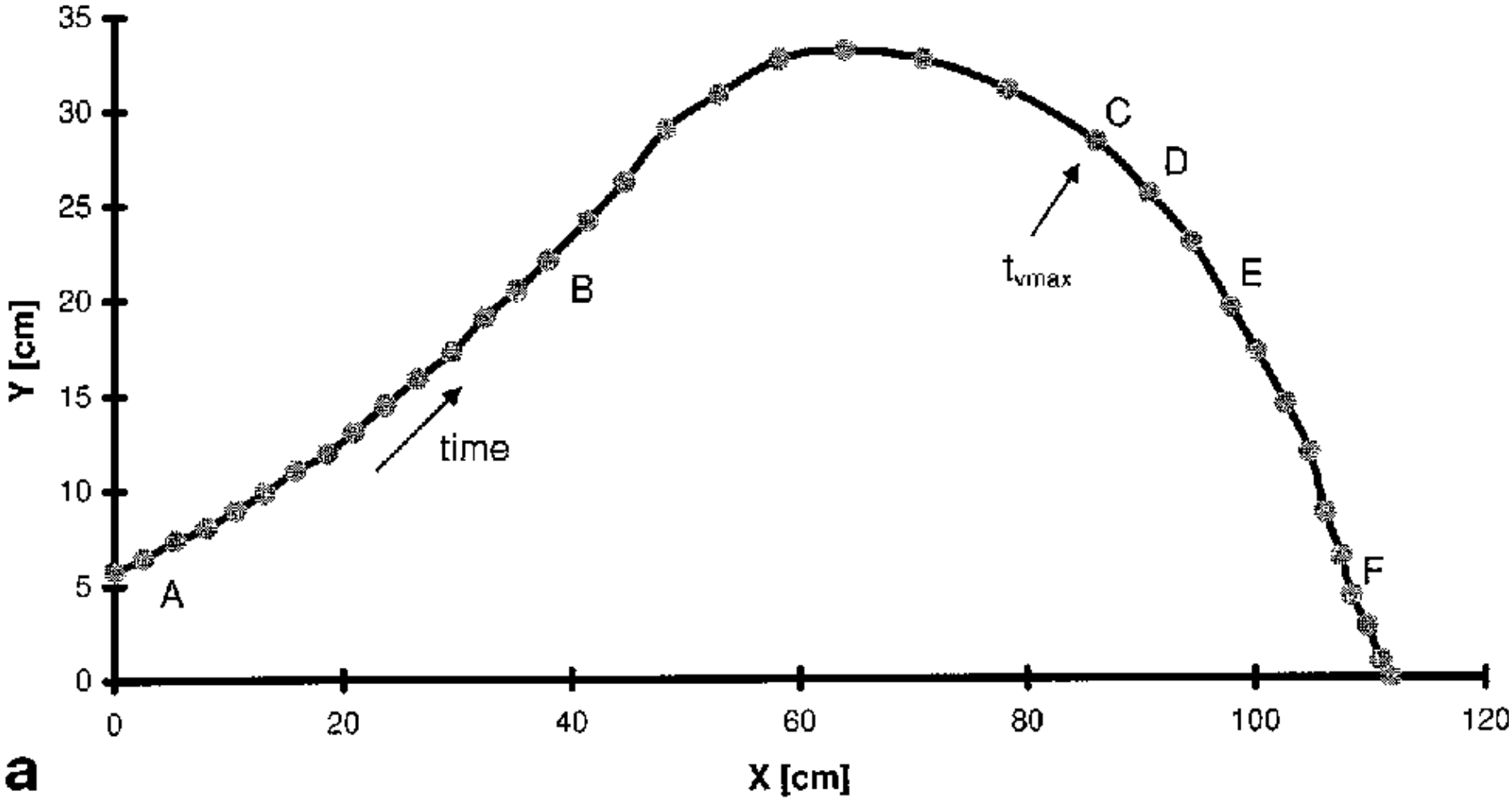


Fig. 5. Experimental setup for simultaneous recording of whip-tip motion and shock wave emission

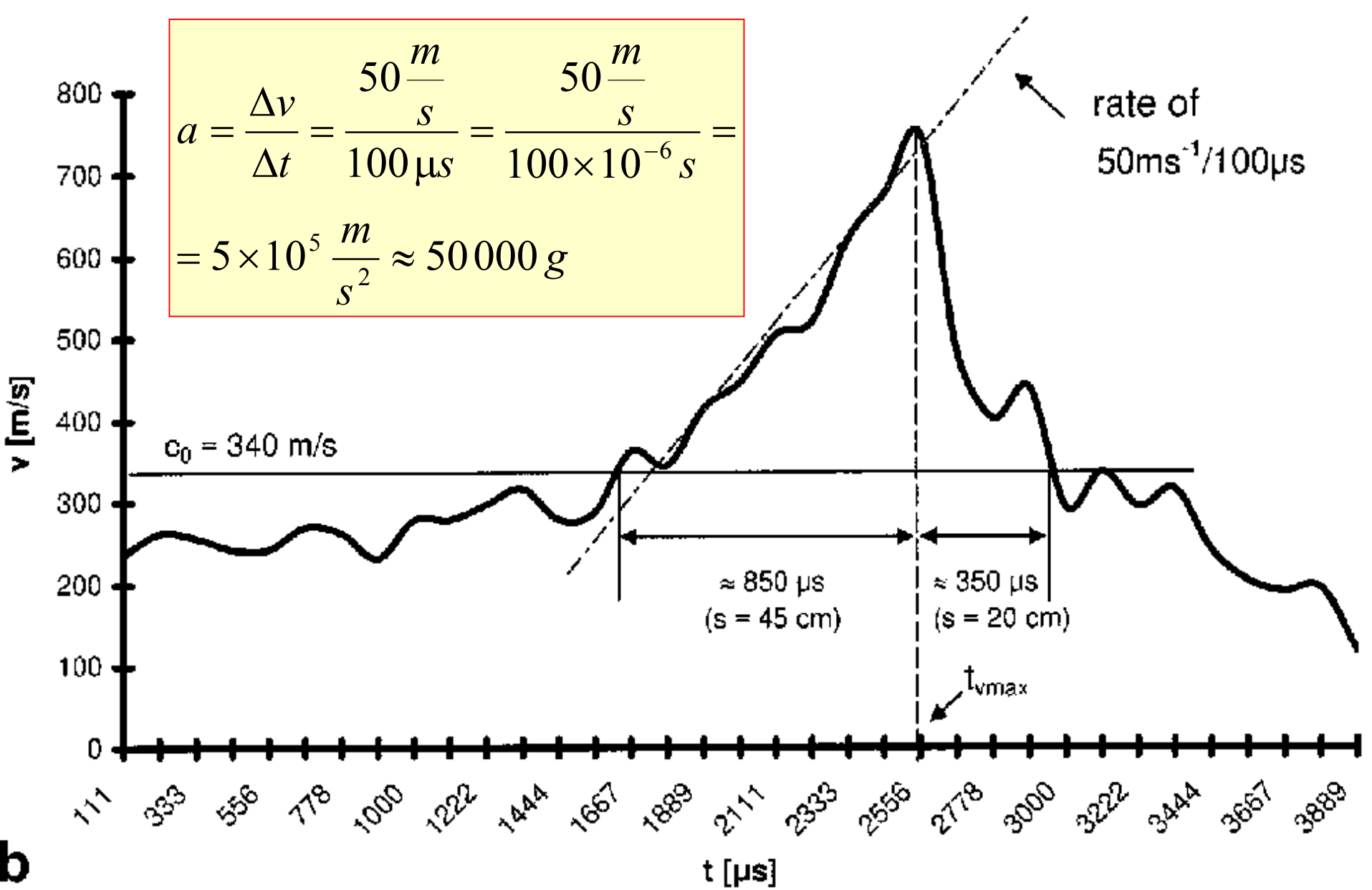

(a)
(b)
(c)
(d)
(e)
(f)
Freiburg


35
30
25
20
15
10
5
0
Y [cm]
0
20
40
60
80
100
120
X [cm]
A
B
C
D
E
F
time
t_{vmax}

a

$$a = \frac{\Delta v}{\Delta t} = \frac{50\,\frac{m}{s}}{100\,\mu s} = \frac{50\,\frac{m}{s}}{100 \times 10^{-6}\,s} =$$

$$= 5 \times 10^{5}\,\frac{m}{s^{2}} \approx 50\,000\,g$$



b

Theory and

numerical simulations

Model matematyczny

W.Tomaszewski

$$x_i = \sum_{j=1}^{i-1} l \cos\varphi_j + \frac{1}{2} l \cos\varphi_i$$

$$y_i = \sum_{j=1}^{i-1} l \sin\varphi_j + \frac{1}{2} l \sin\varphi_i$$

Equations of motions of a set of rods

- 2-nd order Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i}\right) - \frac{\partial \mathcal{L}}{\partial \varphi_i} = 0 \qquad \underbrace{\mathcal{L} = T - U}_{\text{Lagrange function}}$$

J. L. Lagrange (1736-1813)

- Kinetic energy:

$$T = \sum_{i=1}^{n}\left(\overbrace{\frac{1}{2}m\left(\dot{x}_i^2 + \dot{y}_i^2\right)}^{\text{translation}} + \overbrace{\frac{1}{2}I_i\dot{\varphi}_i^2}^{\text{rotation}}\right) = ml^2\sum_{i=1}^{n}\left(\frac{3(n-i)+1}{6}\dot{\varphi}_i^2 + \sum_{j=i+1}^{n}\frac{2(n-i)+1}{2}\dot{\varphi}_i\dot{\varphi}_j\cos(\varphi_i - \varphi_j)\right)$$

- Potential energy:

$$U = \sum_{i=1}^{n}\left(\overbrace{mgy_i}^{\text{gravitation}} + \overbrace{\frac{1}{2}k(\varphi_i - \varphi_{i-1})^2}^{\text{elasticity}}\right) = \sum_{i=1}^{n}\left(\frac{2(n-i)+1)}{2}mgl\sin(\varphi_i) + \frac{1}{2}k(\varphi_i - \varphi_{i-1})^2\right)$$

Examples

- Equations of motion for n=1

$$\frac{1}{3}\frac{d^2\varphi_1}{dt^2}+\frac{1}{2}\frac{g}{l}\cos(\varphi_1)=0$$

- Equations of motion for n=2

$$\frac{4}{3}\frac{d^2\varphi_1}{dt^2}+\frac{1}{2}\frac{d^2\varphi_2}{dt^2}\cos(\varphi_1-\varphi_2)+\frac{1}{2}\left(\frac{d\varphi_2}{dt}\right)^2\sin(\varphi_1-\varphi_2)+\frac{3}{2}\frac{g}{l}\cos(\varphi_1)-k(\varphi_2-\varphi_1)=0,$$

$$\frac{1}{2}\frac{d^2\varphi_1}{dt^2}\cos(\varphi_1-\varphi_2)+\frac{1}{3}\frac{d^2\varphi_2}{dt^2}-\frac{1}{2}\left(\frac{d\varphi_1}{dt}\right)^2\sin(\varphi_1-\varphi_2)+\frac{1}{2}\frac{g}{l}\cos(\varphi_2)-k(\varphi_1-\varphi_2)=0.$$

General form of equations of motion

$$M(\varphi)\ddot{\varphi} + \dot{\varphi}^{\mathsf{T}} M_{\varphi}(\varphi)\dot{\varphi} + C(\varphi) + E\varphi = 0$$

$$\varphi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \quad \dot{\varphi} = \begin{bmatrix} \frac{d\varphi_1}{dt} \\ \vdots \\ \frac{d\varphi_n}{dt} \end{bmatrix} \quad \ddot{\varphi} = \begin{bmatrix} \frac{d^2\varphi_1}{dt^2} \\ \vdots \\ \frac{d^2\varphi_n}{dt^2} \end{bmatrix}$$

General form of equations of motion

$$M(\varphi)=\begin{bmatrix} \frac{3(n-1)+1}{3} & \frac{2(n-1)-1}{2}c_{12} & \frac{2(n-2)-1}{2}c_{13} & \cdots & \frac{1}{2}c_{1n} \\ \frac{2(n-1)-1}{2}c_{12} & \frac{3(n-2)+1}{3} & \frac{2(n-2)-1}{2}c_{23} & \cdots & \frac{1}{2}c_{2n} \\ \frac{2(n-2)-1}{2}c_{13} & \frac{2(n-2)-1}{2}c_{23} & \frac{3(n-3)+1}{3} & \cdots & \frac{1}{2}c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}c_{1n} & \frac{1}{2}c_{2n} & \frac{1}{2}c_{3n} & \cdots & \frac{1}{3} \end{bmatrix}$$

$$C(\varphi)=\frac{g}{l}\begin{bmatrix} \frac{2n-1}{2}c_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2(n-1)-1}{2}c_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{2(n-2)-1}{2}c_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \frac{1}{2}c_n \end{bmatrix}$$

$$E=-k\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$c_i=\cos(\varphi_i),\ c_{ij}=\cos(\varphi_i-\varphi_j),\ s_{ij}=\sin(\varphi_i-\varphi_j)$$

Test:

falling chain

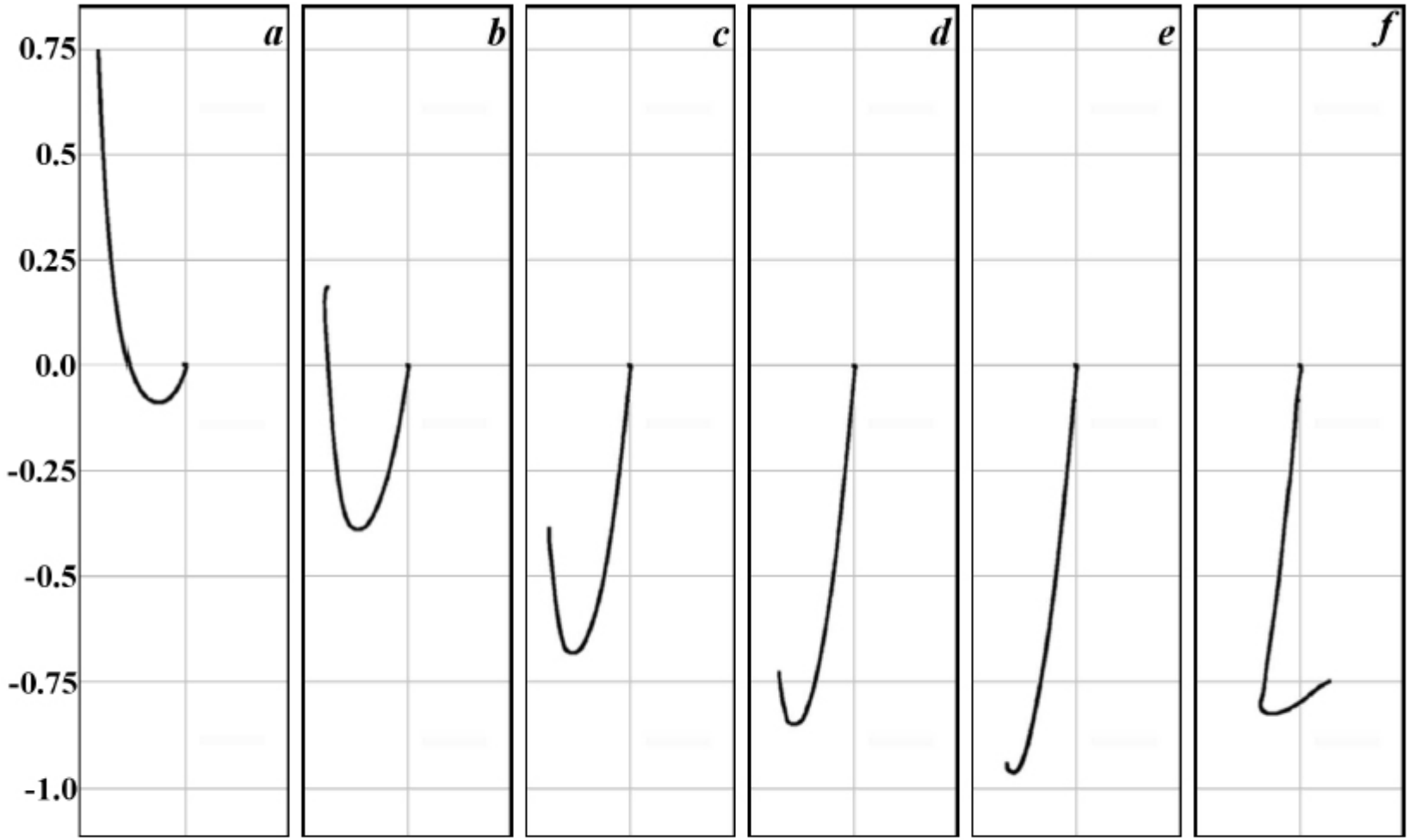


Figure 4. Conformations of the falling chain recorded in the laboratory experiment. Initial condition - broad catenary.

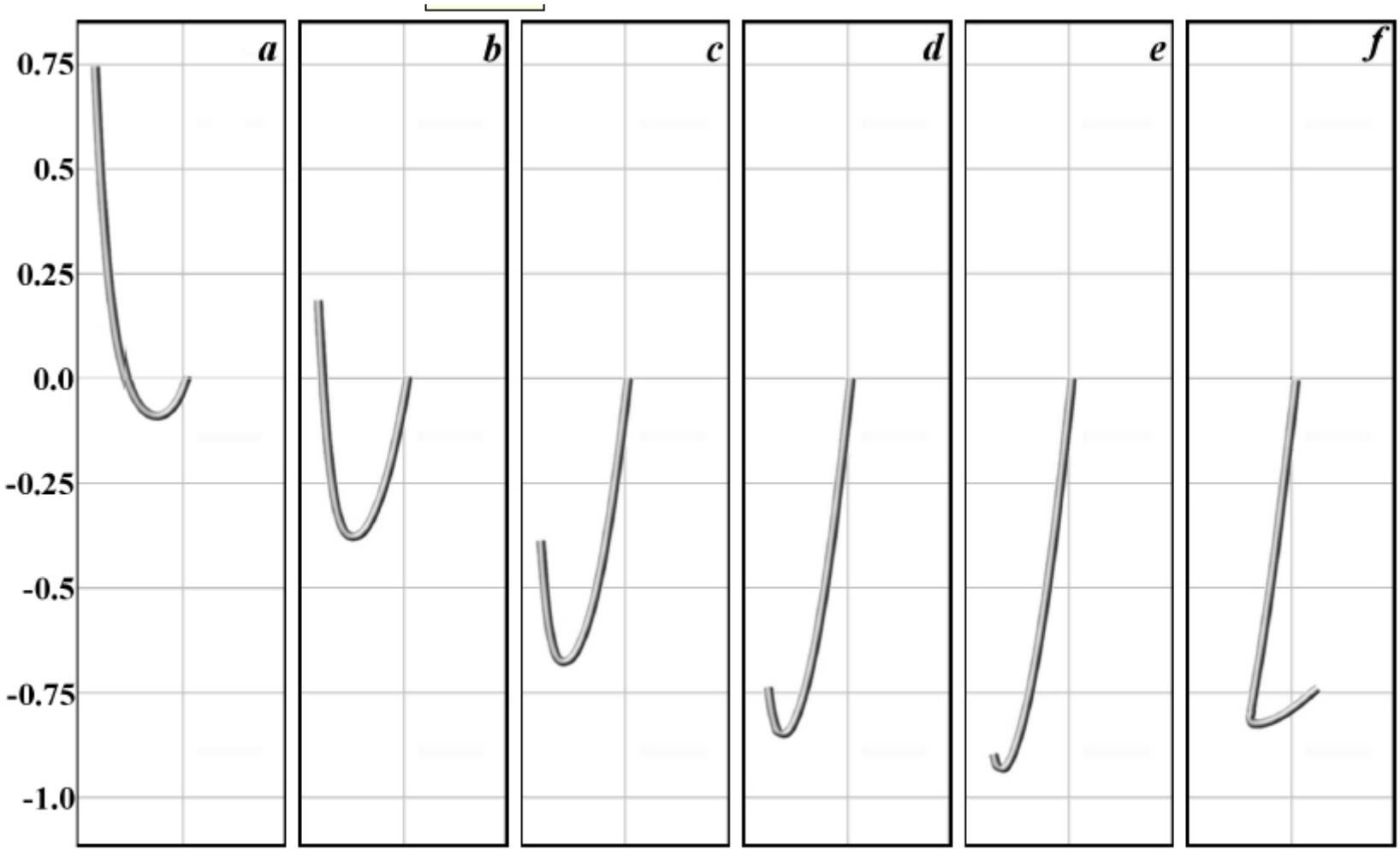
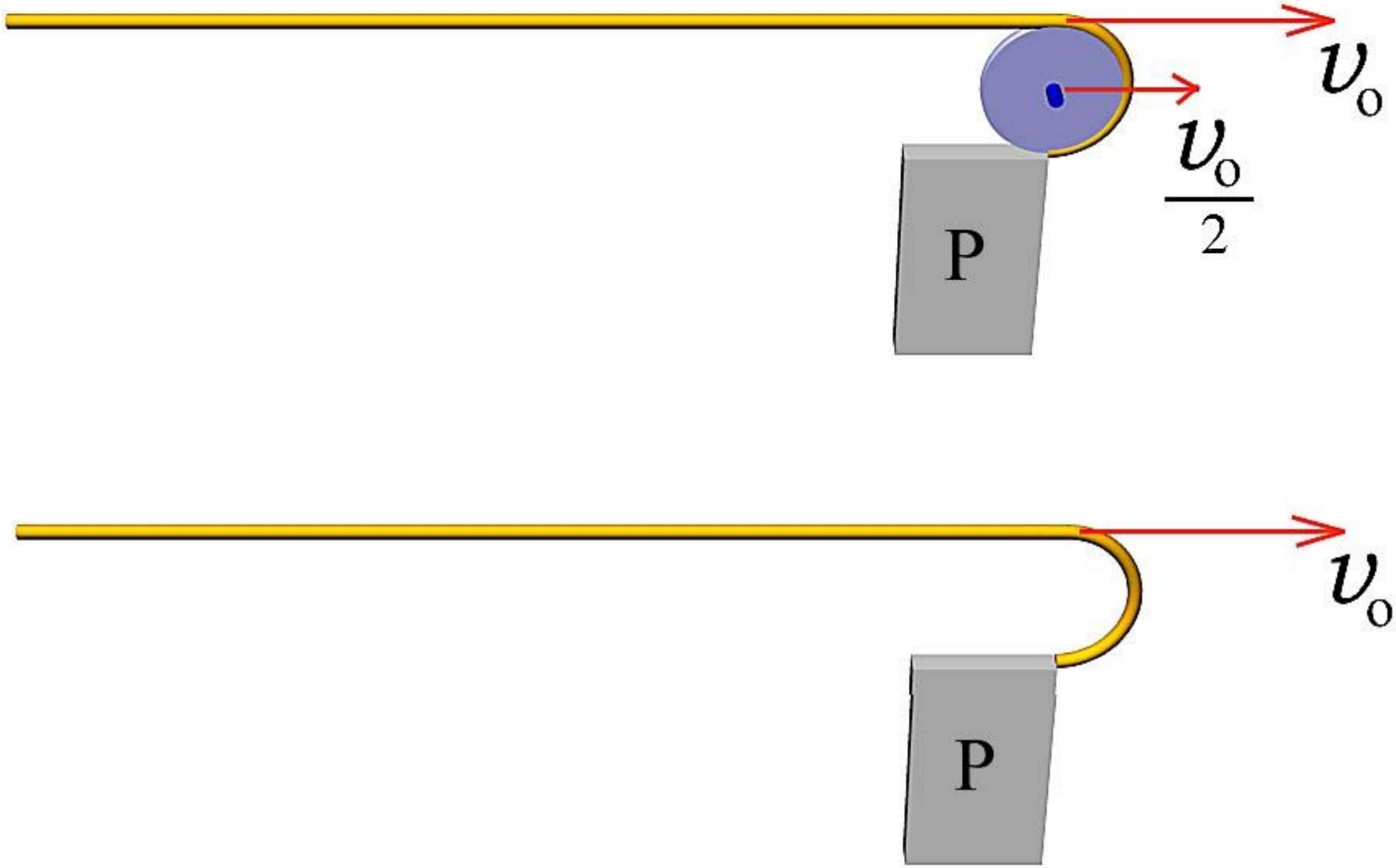
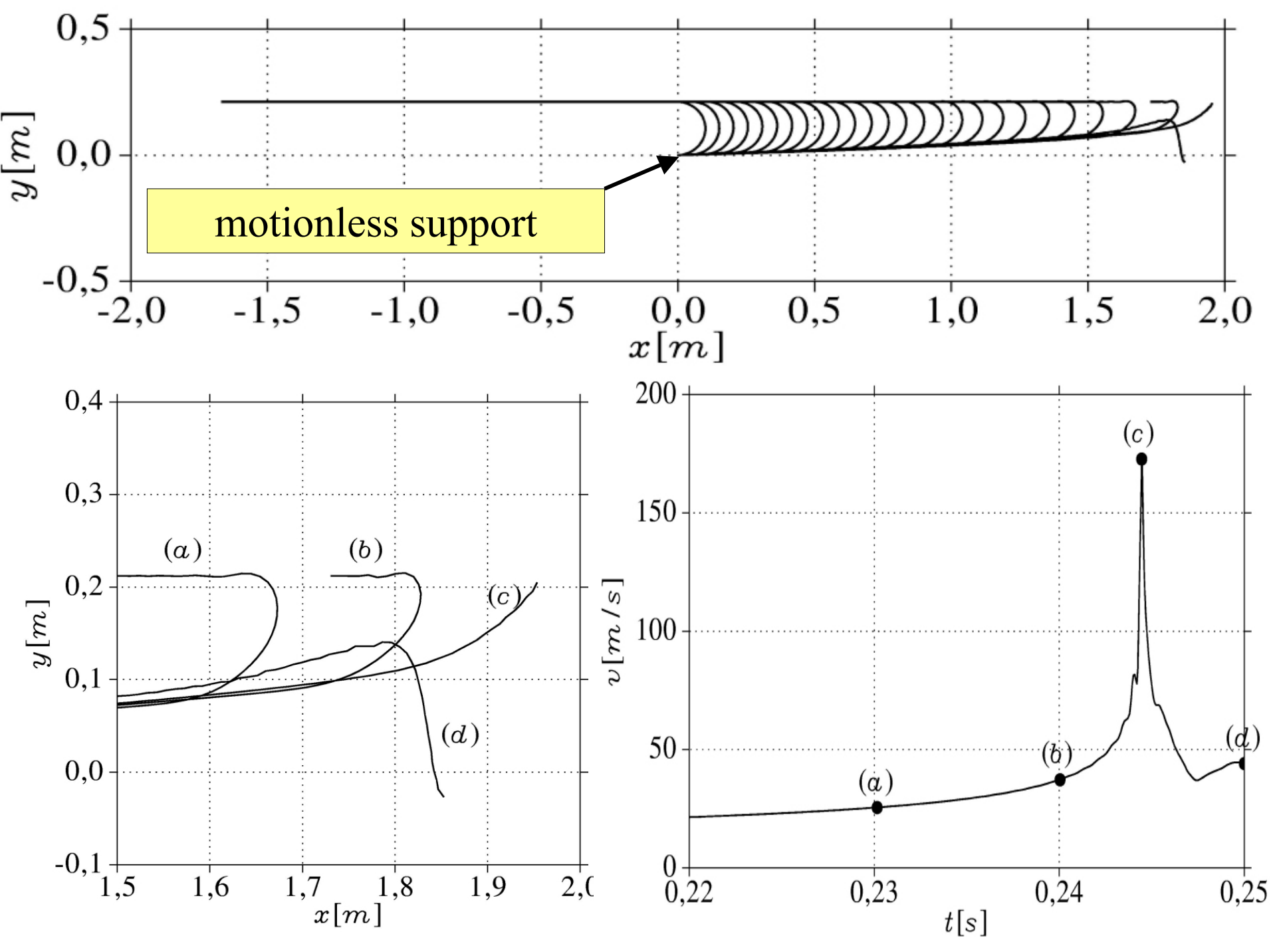
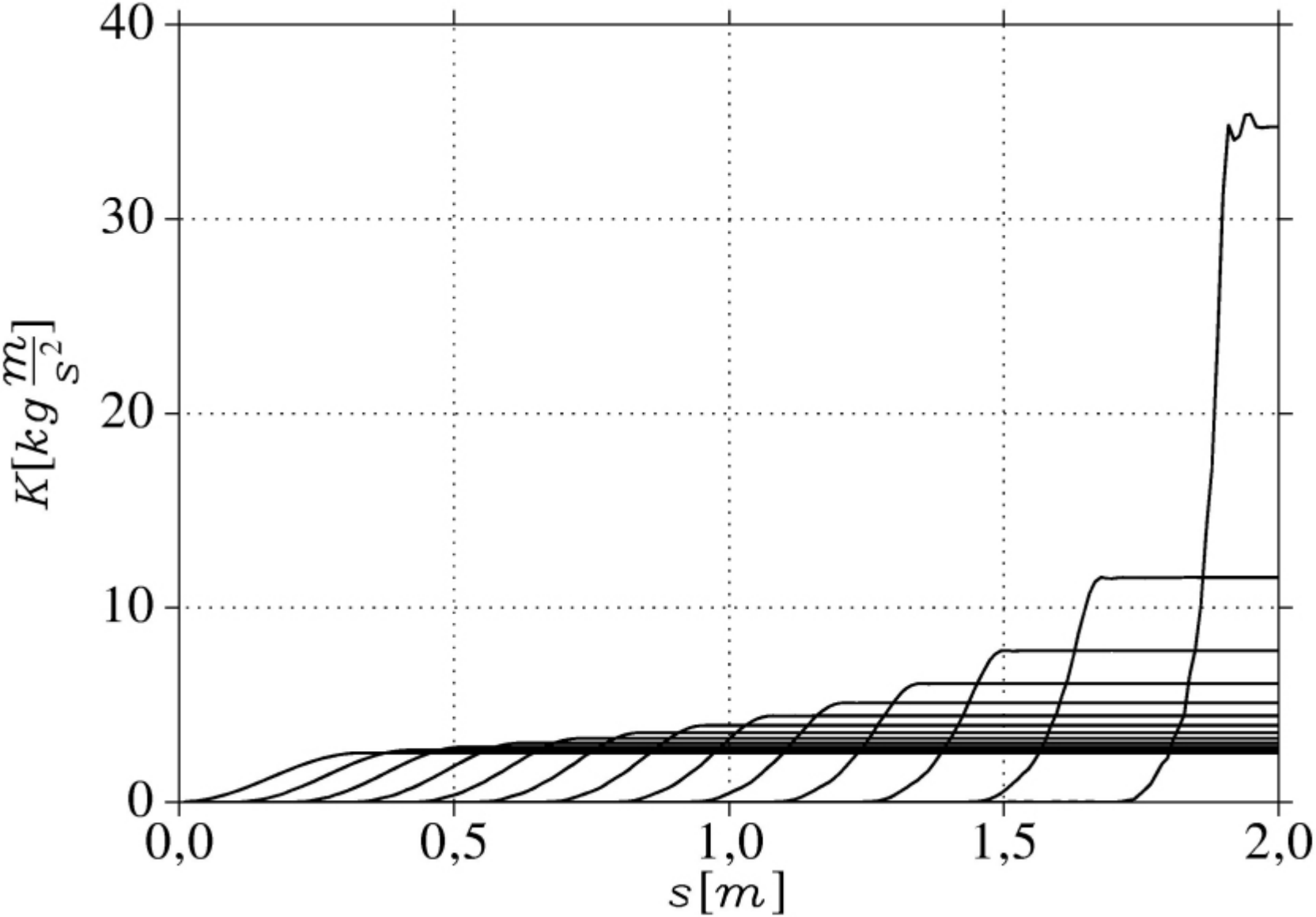


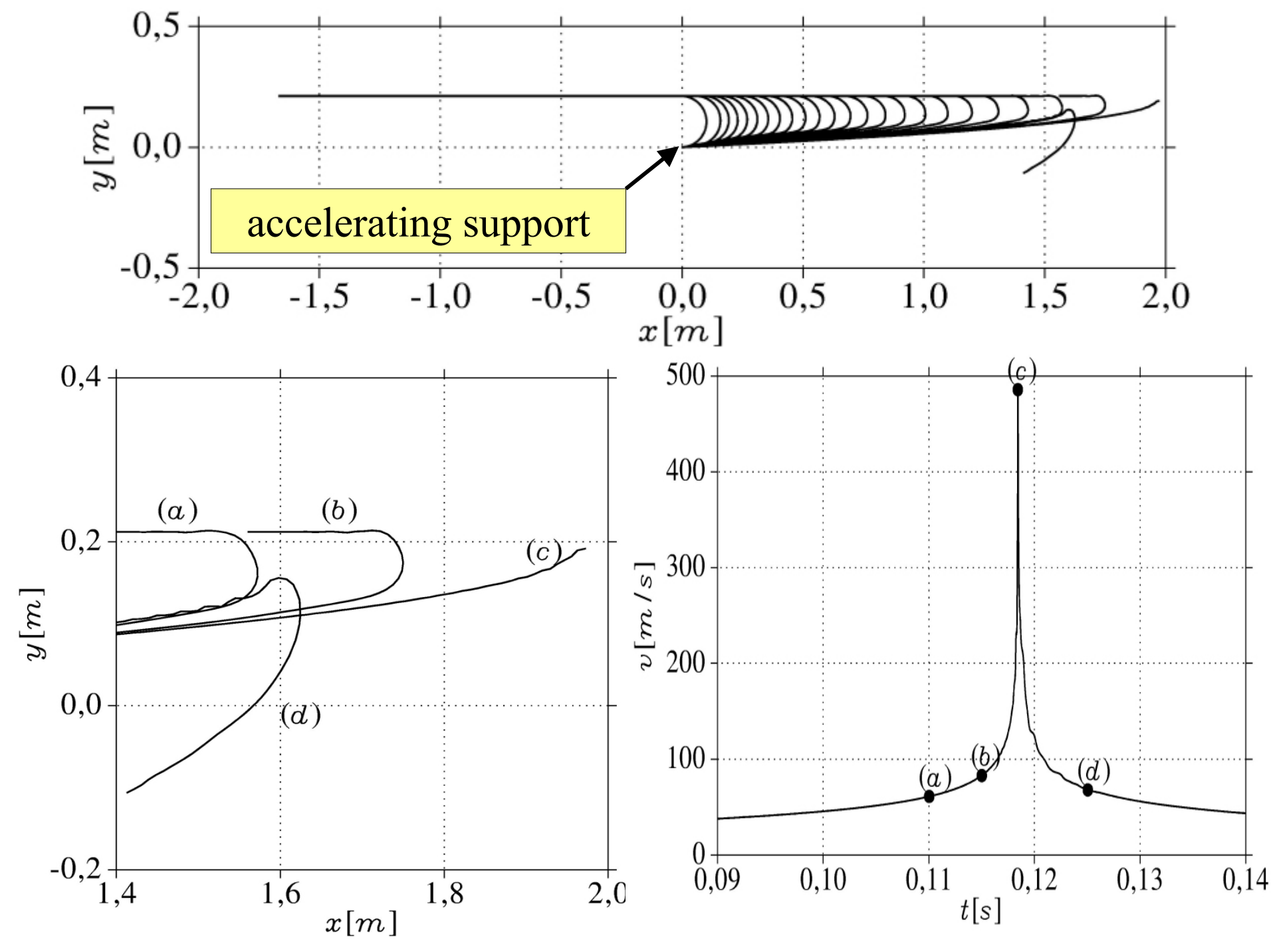
Figure 5. Conformations of the falling chain found in the numerical experiment. Initial condition - broad catenary.

Whip dynamics simulations

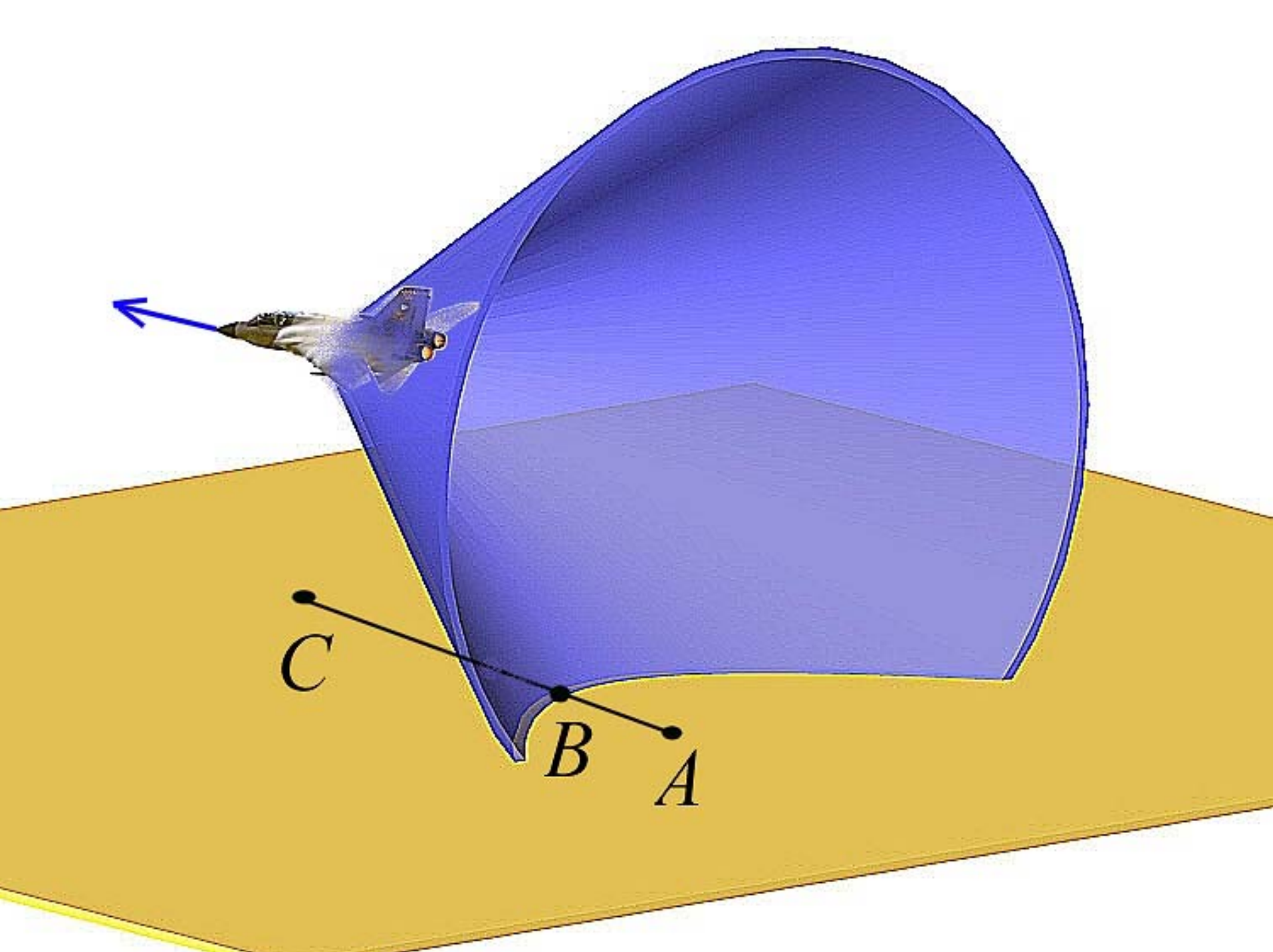

v_o
$\frac{v_o}{2}$
P
v_o
P


0,5
0,0
-0,5
y[m]
-2,0
-1,5
-1,0
-0,5
0,0
0,5
1,0
1,5
2,0
x[m]
motionless support
0,4
0,3
0,2
0,1
0,0
-0,1
y[m]
(a)
(b)
(c)
(d)
1,5
1,6
1,7
1,8
1,9
2,0
x[m]
200
150
100
50
0
v[m/s]
(a)
(b)
(c)
(d)
0,22
0,23
0,24
0,25
t[s]

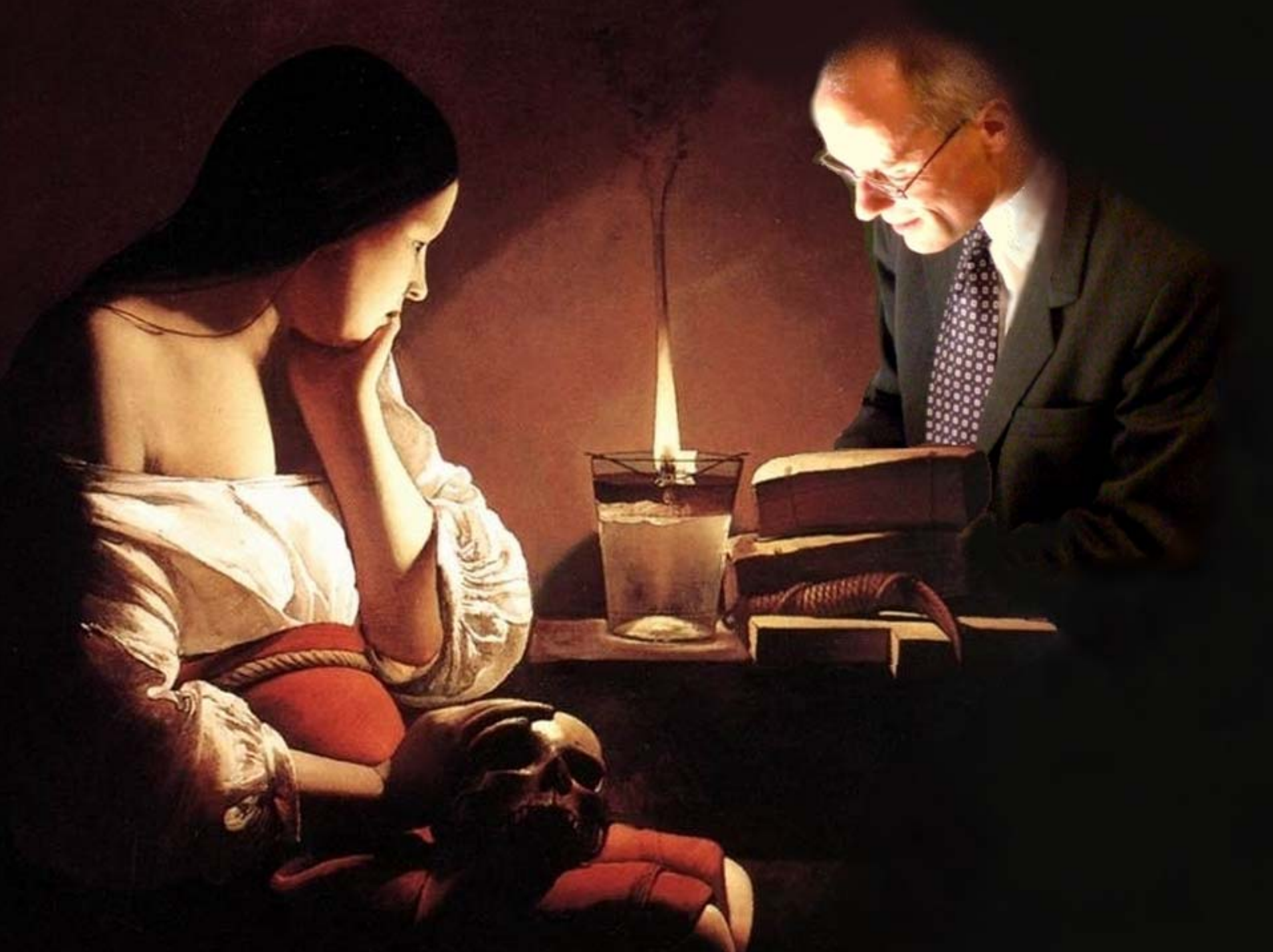

40
30
20
10
0
$K[kg\frac{m}{s^2}]$
0,0
0,5
1,0
1,5
2,0
$s[m]$


accelerating support
0,5
0,0
-0,5
-2,0
-1,5
-1,0
-0,5
0,0
0,5
1,0
1,5
2,0
y[m]
x[m]
0,4
0,2
0,0
-0,2
1,4
1,6
1,8
2,0
(a)
(b)
(c)
(d)
y[m]
x[m]
500
400
300
200
100
0
0,09
0,10
0,11
0,12
0,13
0,14
v[m/s]
t[s]

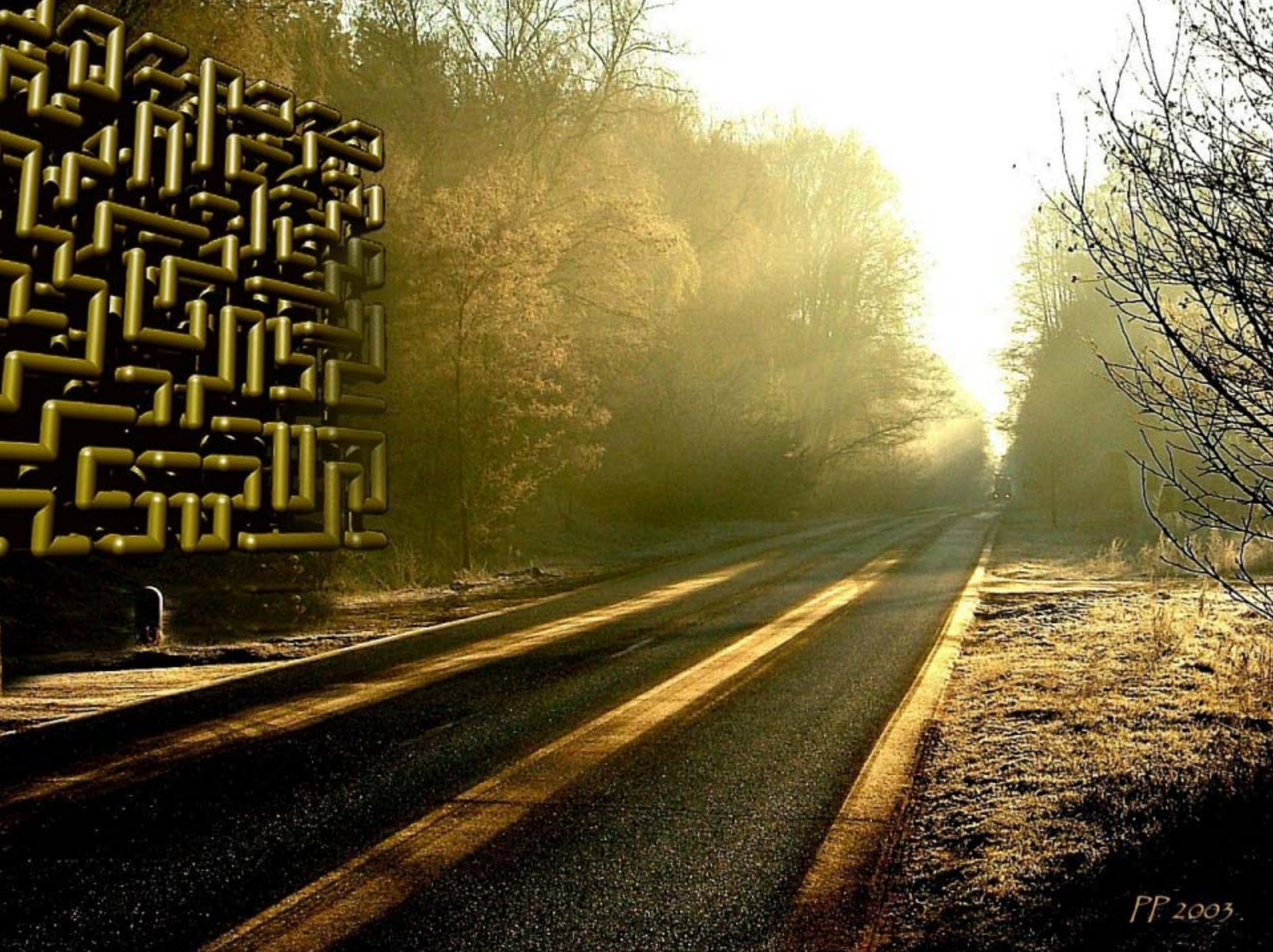
Shock waves


C
B
A

Physics around us




http://liftoff.msfc.nasa.gov/News/2000/News-Flames.asp


PP 2003


VC 511


Pieranie