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MAGNETIC MATERIALS AND HYSTERESIS

Magnets and other sources of magnetic field

Poznań 2019

Maciej Urbaniak

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Magnets and other sources of magnetic field

- Magnetic scalar potential
- Magnetic charges
- Permanent magnets
- Special sources of magnetic field

Magnetic scalar potential

In many practical applications it can be assumed that the magnetization of the body is constant (in value and direction) in weak magnetic fields [1]. We have then (from L1*)

$$\nabla\cdot\vec{B}=0 \qquad\qquad \vec{B}=\mu_0(\vec{H}+\vec{M})$$

It follows that:

$$\nabla\cdot\%imi_0(\vec{H}+\vec{M})=0 \rightarrow \nabla\cdot\vec{H}=-\nabla\cdot\vec{M}$$

If there are no free currents (zero current density**) we can define magnetic scalar potential:

$$\boxed{\vec{H}=-\nabla\varphi} \qquad\qquad \vec{B}(\vec{r})=\nabla\times\vec{A}(\vec{r})$$

Substituting this into the previous equation gives:

$$-\nabla\cdot\nabla\varphi=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi=-\nabla^2\varphi=-\nabla\cdot\vec{M}$$

Poisson's equation:

$$\boxed{\nabla^2\varphi=-\frac{\rho}{\varepsilon_0}}$$

Comparing $\nabla^2\varphi=\nabla\cdot\vec{M}$ with Poisson's equation we can formally introduce magnetic charges:

$$\boxed{\rho_{magn}=-\nabla\cdot\vec{M}}$$

** $\nabla\times\vec{H}=\vec{j}_{free}$

$$\nabla\times\nabla f(x,y,x)=0$$

f - scalar function

* Lecture no. 1

Magnetic scalar potential

We have as a solution to Poisson's equation $\nabla^2\varphi=\nabla\cdot\vec{M}$ [4]:

$$\varphi(\vec{r})=-\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'$$

$$(ab)'=ab'+a'b$$

Using the expression for a derivative of a product (integrating by parts) we obtain:

$$\varphi(\vec{r})=-\frac{1}{4\pi}\int_V \nabla'\cdot\left(\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right)d^3r'+\frac{1}{4\pi}\int_V \vec{M}(\vec{r}\,')\cdot\nabla'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)d^3r'$$

And by the Gauss' theorem for the first term we have:

$$\varphi(\vec{r})=\frac{1}{4\pi}\int_{far\,surface}\frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\boldsymbol{S}'-\frac{1}{4\pi}\int_V \vec{M}(\vec{r}\,')\cdot\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)d^3r'$$

We change sign of the second term using:

$$\nabla\left(\frac{1}{|r-r'|}\right)=-\nabla'\left(\frac{1}{|r-r'|}\right)$$

Since the magnetization vanishes at infinity (we seek the potential of bounded magnetization) the first integral vanishes. In the second integral we move ***M*** under nabla as ***M*** does not depend on ***r***. Finally we get:

$$\boxed{\varphi(\vec{r})=-\frac{1}{4\pi}\nabla\cdot\int_V \frac{\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'}$$

Magnetic charges – coulombian approach* (from C.A. de Coulomb)

We have (from L1*) the induction of magnetic dipole. We are looking for the magnetic scalar potential that gives the field $\mu_0 H$ of the dipole [2].

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{|\vec{r}|^3}$$

μ_0 is actually not a "part" of φ

Setting:

$$\phi(\vec{r})=\boxed{\frac{\mu_0}{4\pi}\vec{m}\cdot\nabla'\frac{1}{|\vec{r}|}}=\frac{\mu_0}{4\pi}\vec{m}\cdot\nabla'\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)=$$

$$\frac{\mu_0}{4\pi}\vec{m}\cdot\left(\frac{\vec{r}}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\right)=\frac{\mu_0}{4\pi}\left(\frac{(x-x')m_x+(y-y')m_y+(z-z')m_z}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}}\right)$$

and differentiating (with respect to unprimed coordinates) we get:

$$\mu_0\vec{H}=-\mu_0\nabla\phi(\vec{r})=\frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{|\vec{r}|^3}$$

Integrating over volume containing magnetic moments we get the potential of the dipole distribution:

$$\phi_m(\vec{r})=\frac{1}{4\pi}\int\vec{M}\cdot\nabla'\frac{1}{|\vec{r}|}d^3r'$$

, where $\vec{M}\,d^3r'$ is the moment of infinitesimal volume

*[12] H. L. Rakotoarison, J-P Yonnet, B. Delinchant, IEEE Transactions on Magnetics **43**, 1261 (2007)

Magnetic charges

Using now the identity $\nabla\cdot(f\vec{a})=\vec{a}\cdot\nabla f+f\,\nabla\cdot\vec{a}$ [11] we get:

$$\vec{M}\cdot\nabla'\frac{1}{|\vec{r}|}=\nabla'\cdot\left(\frac{1}{|\vec{r}|}\vec{M}\right)-\frac{1}{|\vec{r}|}\nabla'\cdot\vec{M}$$

Inserting this into the integral from the previous page we can rewrite:

$$\phi_m(\vec{r})=\int\vec{M}\cdot\nabla'\frac{1}{|\vec{r}|}d^3r'=\int\nabla'\cdot\left(\frac{1}{|\vec{r}|}\vec{M}\right)d^3r'-\int\frac{1}{|\vec{r}|}\nabla'\cdot\vec{M}\,d^3r'$$

From Gauss' theorem we get: $\int\nabla'\cdot\left(\frac{1}{|\vec{r}|}\vec{M}\right)d^3r'=\oint\frac{\vec{M}}{|\vec{r}|}\vec{ds}$

Finally, for the magnetic scalar potential of the bounded dipole distribution, we obtain:

$$\boxed{\phi_m(\vec{r})=\frac{1}{4\pi}\oint_S\frac{\vec{M}\cdot\vec{ds}}{|\vec{r}|}-\frac{1}{4\pi}\int_V\frac{\nabla'\cdot\vec{M}}{|\vec{r}|}d^3r'}$$

Magnetic charges

Using now the identity $\nabla\cdot(f\vec{a})=\vec{a}\cdot\nabla f+f\nabla\cdot\vec{a}$ [11] we get:

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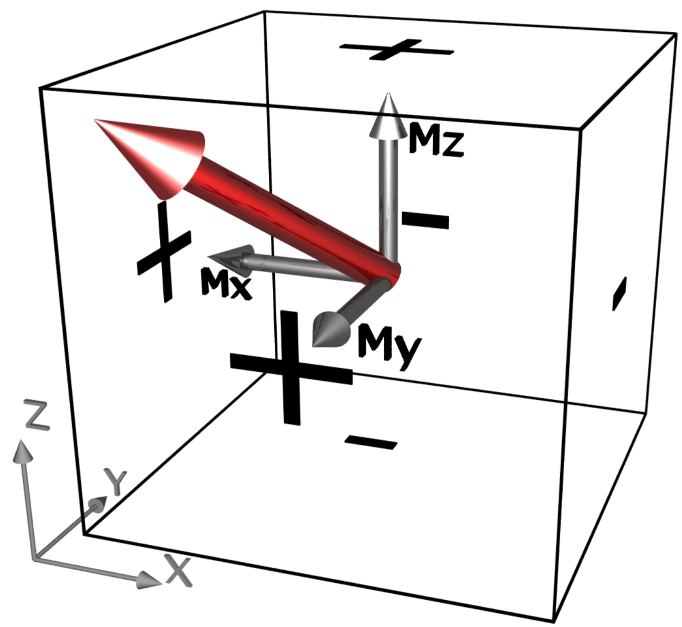
$$\phi_m(\vec{r})=\boxed{\frac{1}{4\pi}\oint_S\frac{\vec{M}\cdot\vec{ds}}{|\vec{r}|}}-\boxed{\frac{1}{4\pi}\int_V\frac{\nabla'\cdot\vec{M}}{|\vec{r}|}d^3r'}$$

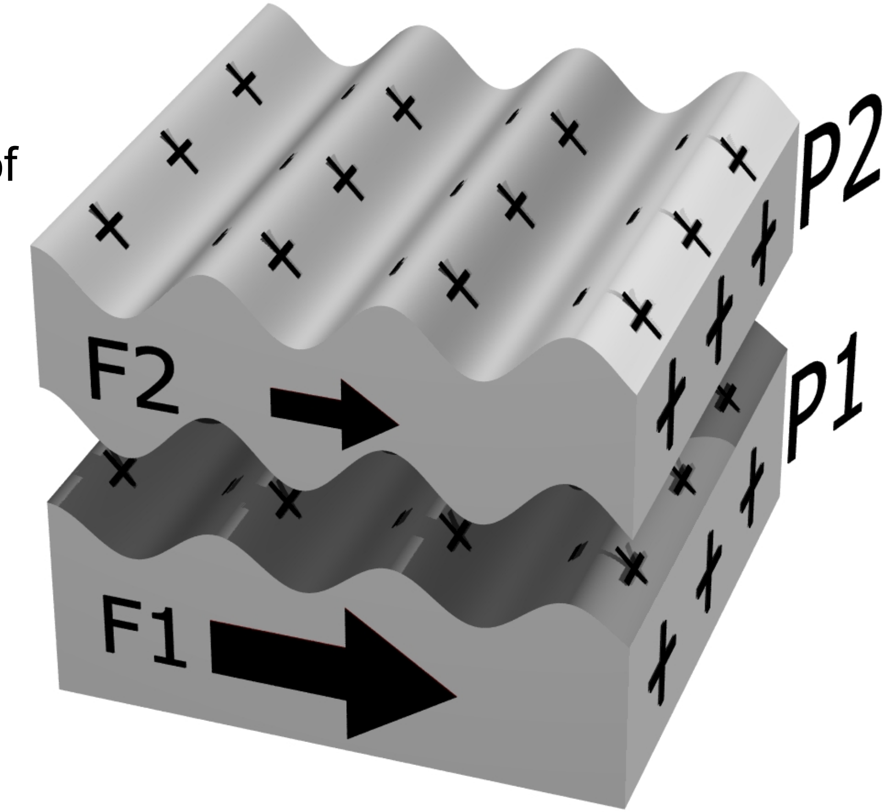
$-\nabla\cdot\vec{M}$ volume magnetic charges

$\vec{M}\cdot\vec{ds}$ surface magnetic charges – magnetic poles



Magnetic charges

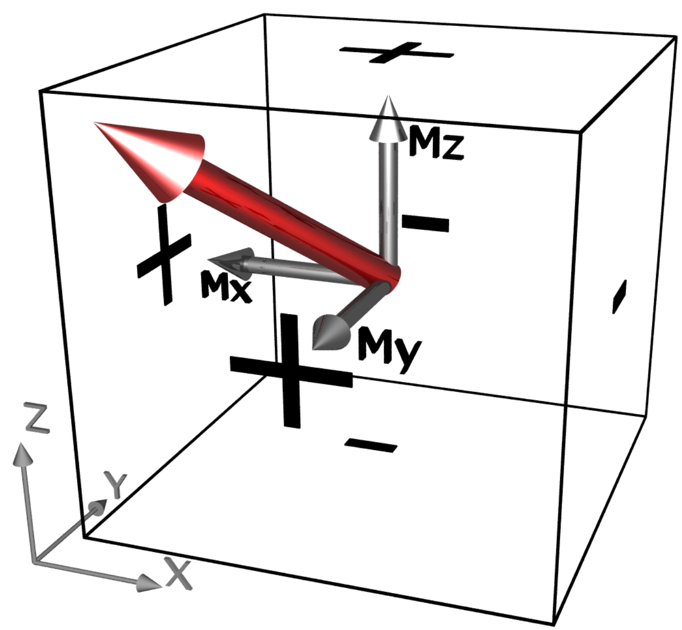
Example from the magnetostatic coupling of thin magnetic films (Neél's coupling):



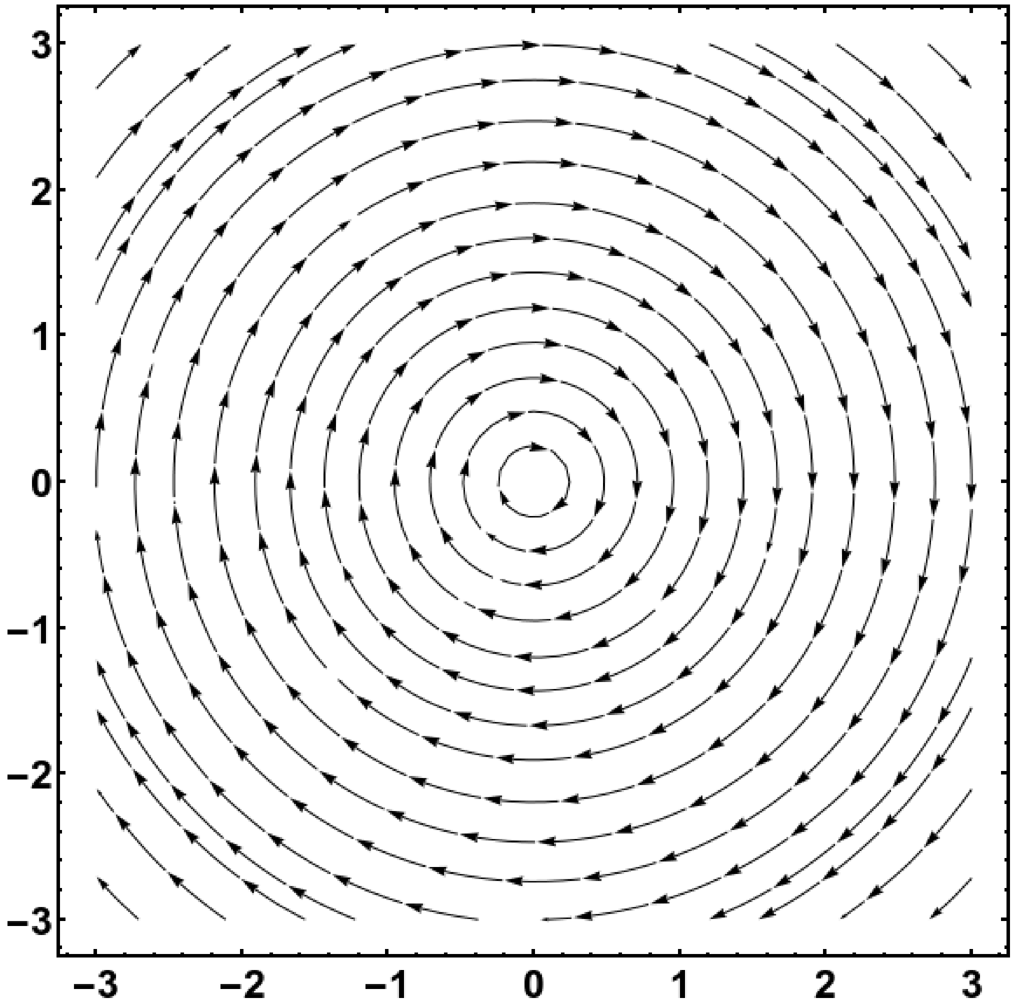
$$\phi_m(\vec{r}) = \boxed{\frac{1}{4\pi}\oint_S \frac{\vec{M}\cdot\vec{ds}}{|\vec{r}|}} - \boxed{\frac{1}{4\pi}\int_V \frac{\nabla'\cdot\vec{M}}{|\vec{r}|}d^3r'}$$

$-\nabla\cdot\vec{M}$ volume magnetic charges

$\vec{M}\cdot\vec{ds}$ surface magnetic charges – magnetic poles



Magnetic volume charges

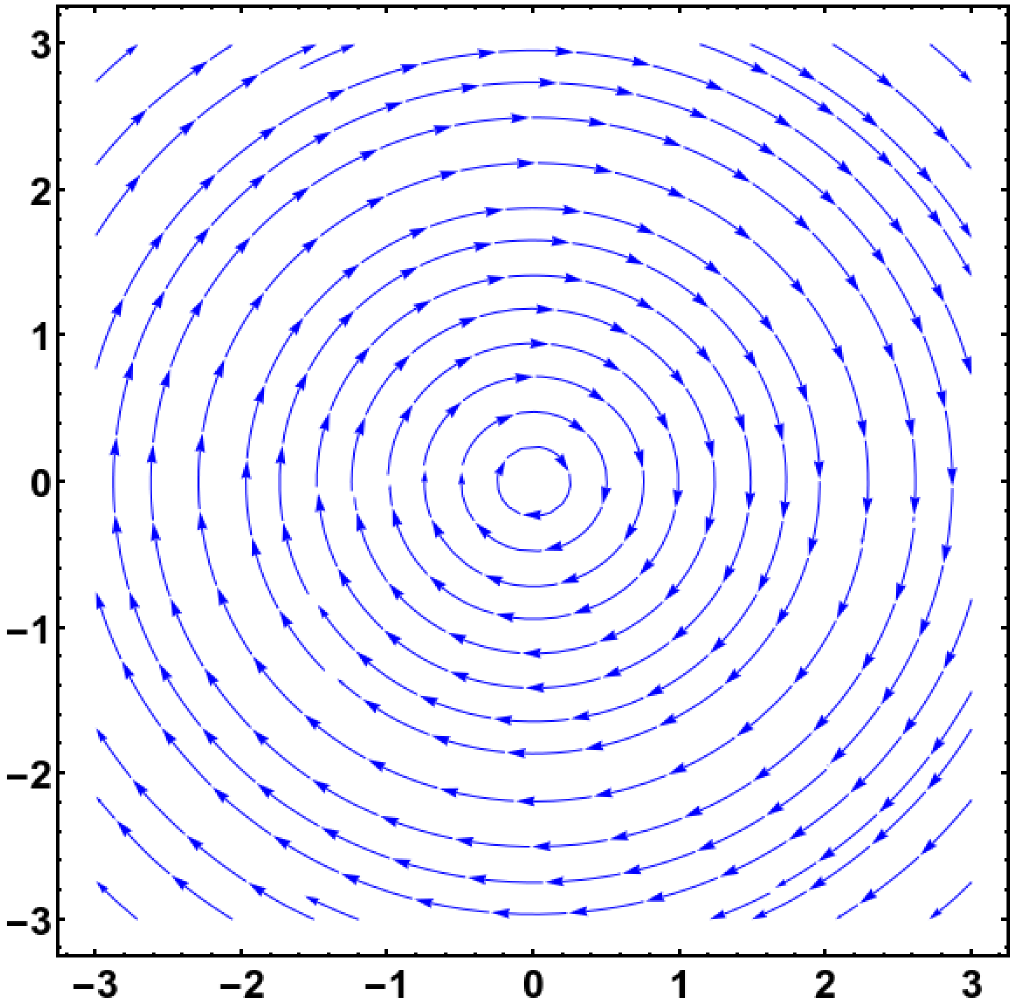


$$\vec{M} = \left(\frac{y}{\sqrt{x^2 + y^2}}, -\frac{x}{\sqrt{x^2 + y^2}} \right)$$

no volume (2D example) charges

Constant magnitude **divergenceless** vector field

Magnetic volume charges

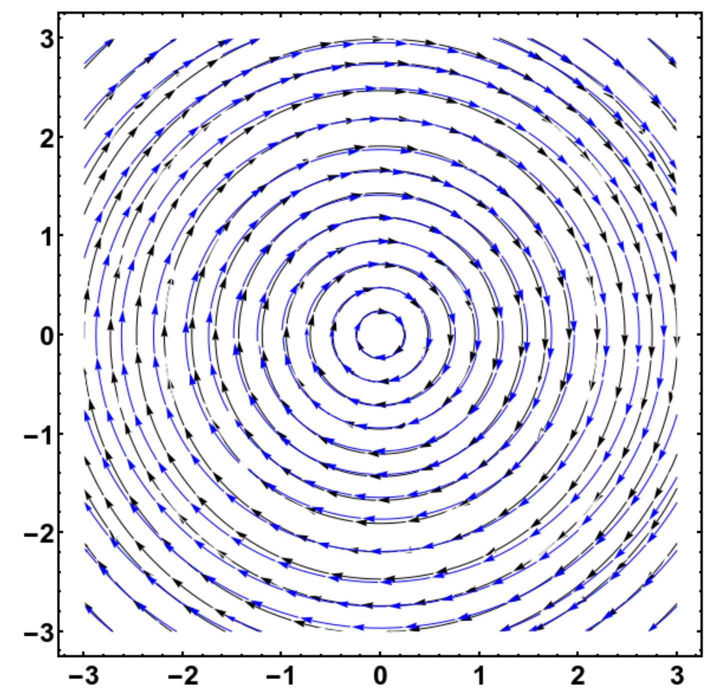


$$\vec{M} = \left(\frac{1.1\ y}{\sqrt{x^2 + 1.21\ y^2}}, -\frac{x}{\sqrt{x^2 + 1.21\ y^2}} \right)$$

$$\nabla \cdot \vec{M} = \frac{0.11\, x\, y}{(x^2 + 1.21\ y^2)^{1.5}}$$

volume charges present

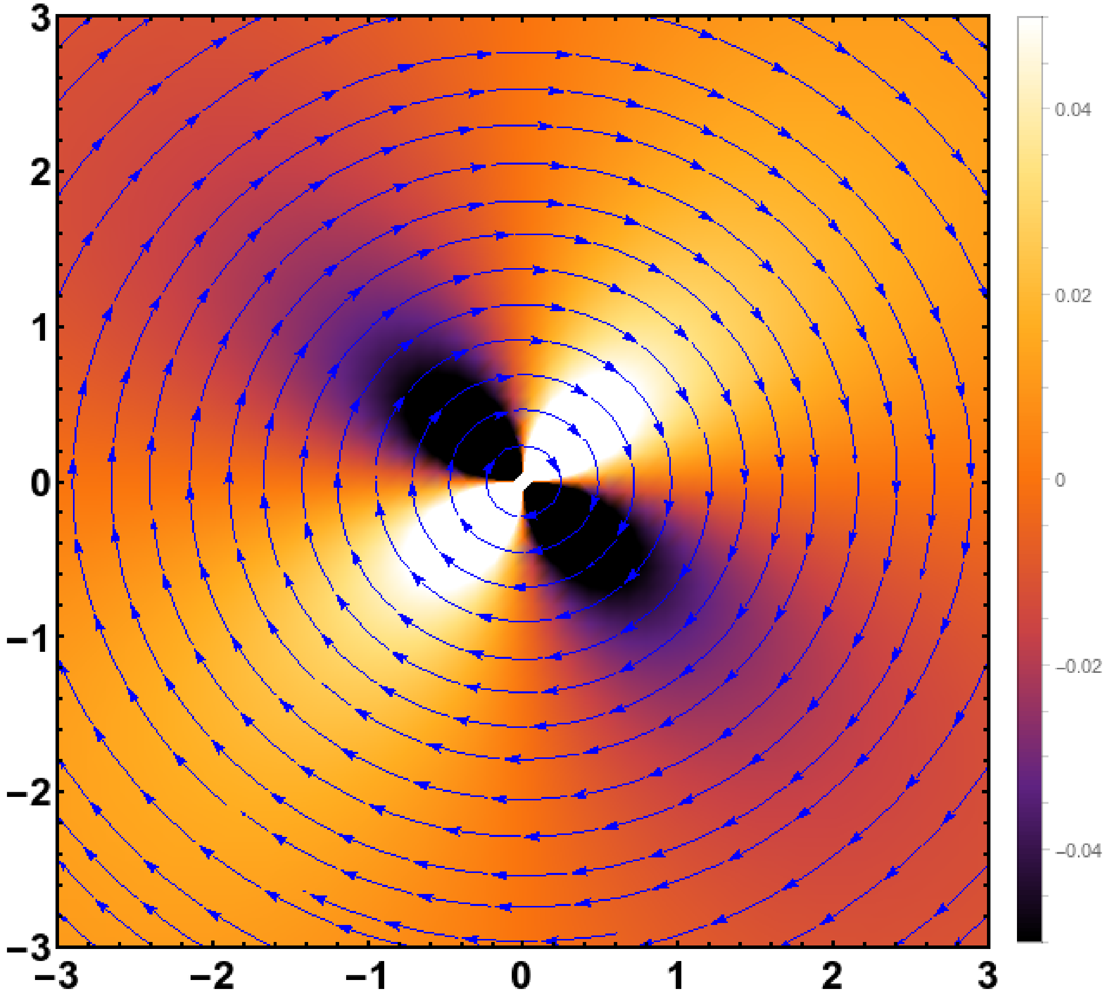
Left plot combined with the one from previous page



Constant magnitude vector field **with divergence**

Visual inspection of magnetic vector fields is of limited use.

Magnetic volume charges



$$\vec{M} = \left(\frac{1.1\ y}{\sqrt{x^2+1.21\ y^2}}, -\frac{x}{\sqrt{x^2+1.21\ y^2}} \right)$$

$$\nabla \cdot \vec{M} = \frac{0.11\ x\ y}{(x^2+1.21\ y^2)^{1.5}}$$

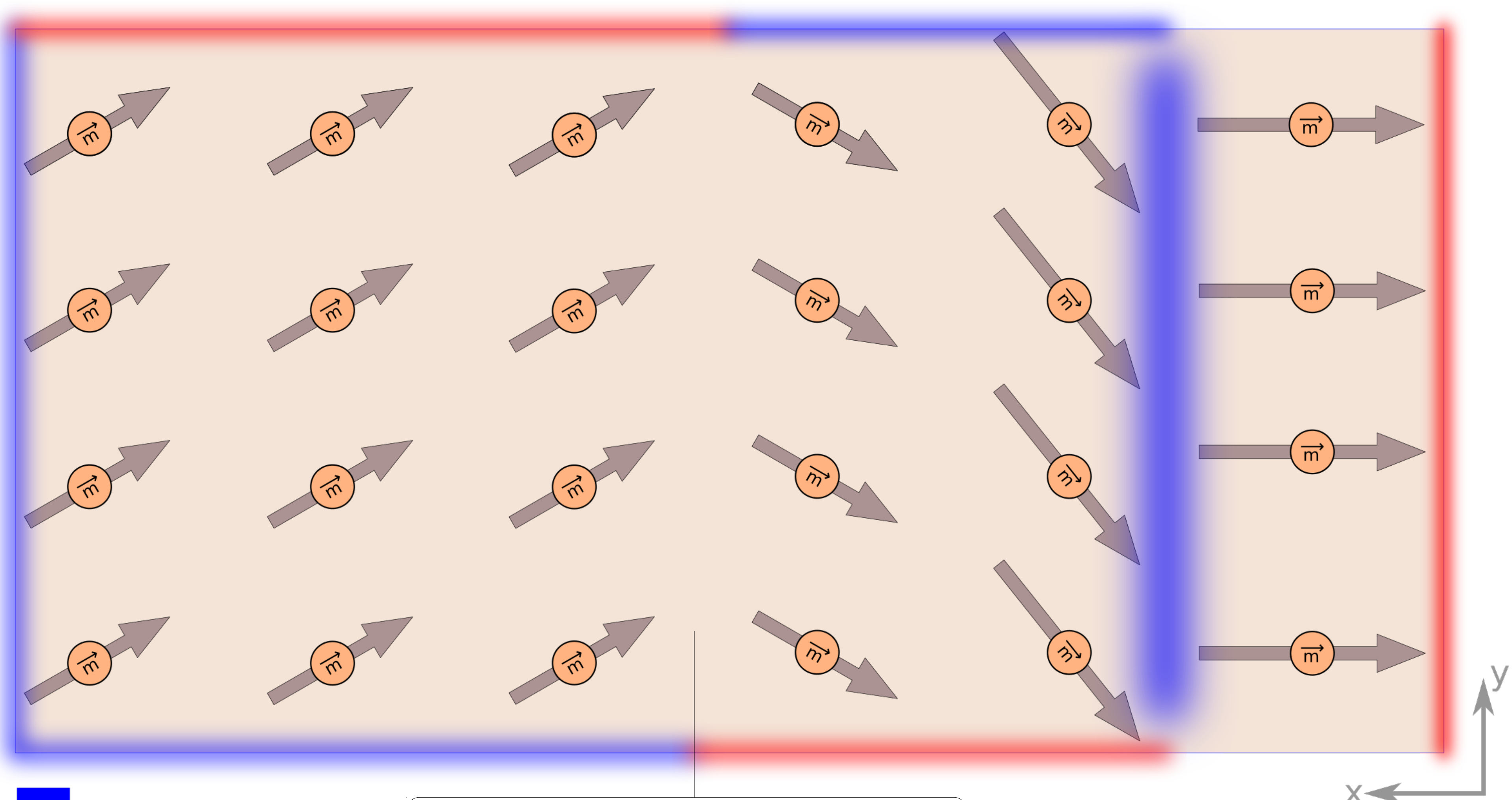
volume charges present

magnetic quadrupole

Constant magnitude vector field **with divergence**

Visual inspection of magnetic vector fields is
of limited use.

Magnetic volume and surface charges



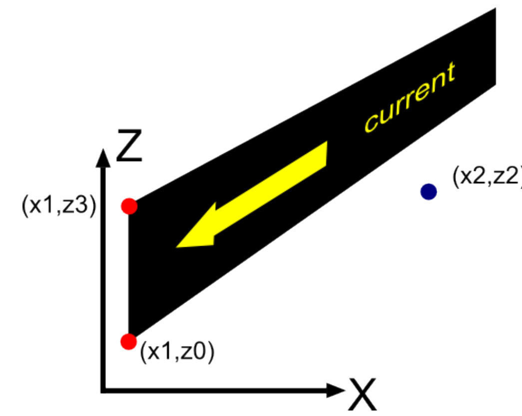
$$\frac{\partial M_x}{\partial x}=0 \qquad \frac{\partial M_y}{\partial y}=0 \rightarrow \nabla\cdot\vec{M}=0 \qquad (M_z=const)$$

- Change of the direction or the magnitude of the magnetization is not necessarily accompanied by the presence of magnetic charges

Magnetic field of current sheet

Current sheet is an extension of current line (i.e. wire). It is a set of current lines bunched together to form a conducting stripe. We have from L.1 for the induction of straight wire carrying current *I*:

$$\vec{B}(R)=\frac{\mu_0 I}{2\pi R}$$



Integrating (the sheet extends from -∞ to +∞ along y-axis) we get:

$$B_z=\frac{\mu_0 K}{2\pi}\left[\arctan\left(\frac{z_2-z}{x_2-x_1}\right)\right]_{z0}^{z3}, \quad \text{K- surface current density}$$

$$B_x=\frac{\mu_0 K}{2\pi}\left[\ln\left[(x_2-x_1)^2-(z_2-z)^2\right]\right]_{z0}^{z3}$$

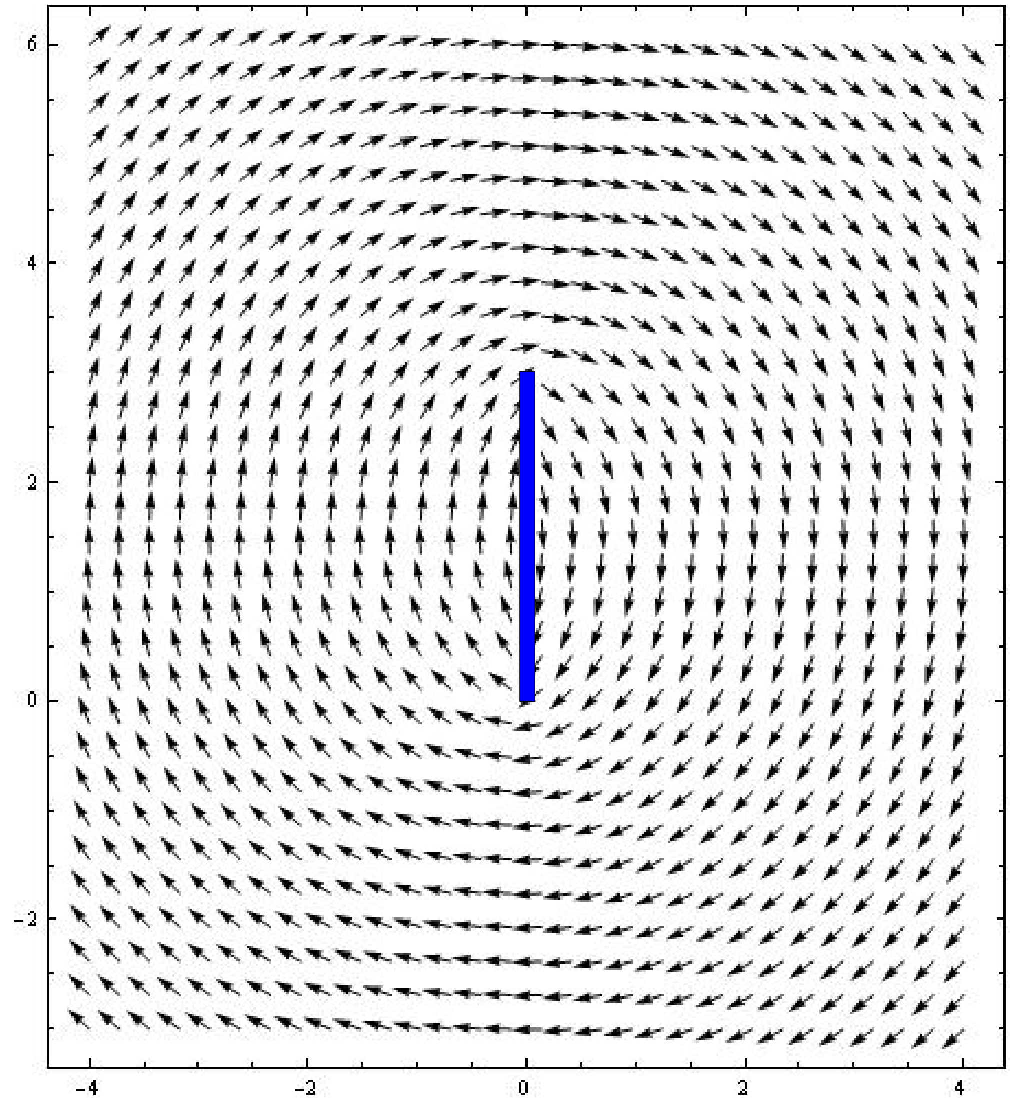
Magnetic field of current sheet

For the current sheet we have:

$$B_z = \frac{\mu_0 K}{2\pi}\left[\arctan\left(\frac{z_2 - z}{x_2 - x_1}\right)\right]_{z0}^{z3}$$

$$B_x = \frac{\mu_0 K}{2\pi}\left[\ln\left[(x_2 - x_1)^2 - (z_2 - z)^2\right]\right]_{z0}^{z3}$$

From far out the magnetic field of the current sheet resembles that of the current line



*

the current flows into the image

x1=0, z1=0, z3=3

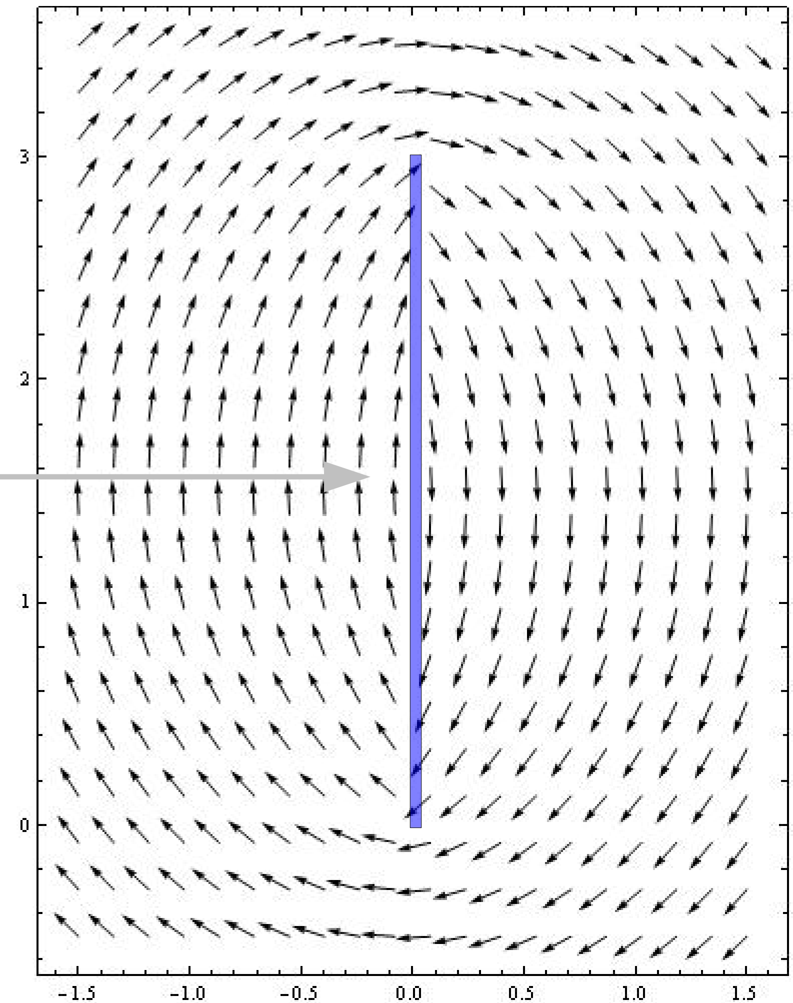
*for better viewing the plot shows only the direction of the field (vectors' length is constant)

Magnetic field of current sheet

In the vicinity of the sheet there is a discontinuity of a tangential component of the magnetic induction **B.**

$$B_z=\frac{\mu_0 K}{2\pi}\left[\arctan\left(\frac{z_2-z}{x_2-x_1}\right)\right]_{z0}^{z3}$$

$$B_x=\frac{\mu_0 K}{2\pi}\left[\ln\left[(x_2-x_1)^2-(z_2-z)^2\right]\right]_{z0}^{z3}$$



*

the current flows into the image

x1=0,z1=0,z3=3

*for better viewing the plot shows only the direction of the field (vectors' length is constant)

Ampere's law

We have from L1: $\nabla \times \vec{B} = \mu_0 \vec{J}(\vec{r})$ *. Using Stoke's theorem we obtain [3]:

$$\oint_{closed\ curve} \vec{B}(\vec{r}) \cdot d\boldsymbol{s} = \int_{bounded\ surface} \nabla \times \vec{B}(\vec{r}) \cdot d\boldsymbol{S} = \int_{bounded\ surface} \mu_0 \vec{J}(\vec{r}) \cdot d\boldsymbol{S} = \mu_0 I$$

Rewriting we find that:

$$\oint_{closed\ curve} \vec{B}(\vec{r}) \cdot d\boldsymbol{s} = \mu_0 I$$

which is *Ampere's Law*

The law can be used for calculating magnetic fields for symmetric current distributions

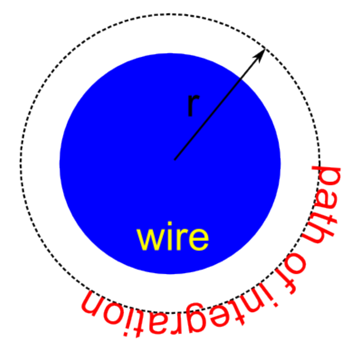
Example: Field within and outside the wire of radius R

a) outside the wire:

$$\oint \vec{B}(\vec{r}) \cdot d\boldsymbol{s} = 2\pi r B \rightarrow B = \frac{\mu_0 I_{total}}{2\pi r}$$

b) inside the wire:

$$I = I_{total} \frac{r^2}{R^2} \rightarrow B = \left(I_{total} \frac{r^2}{R^2} \right) \frac{\mu_0}{2\pi r} = \frac{\mu_0 I_{total}}{2\pi R^2} r$$



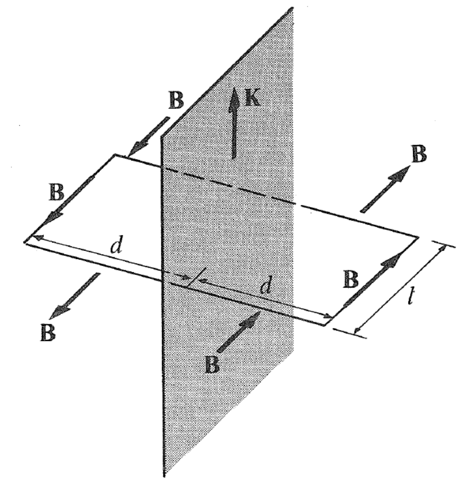
* this is called Ampere's law in differential form [4].

Discontinuity of *B* due surface current

We use Amperes's law to find B discontinuity [3].

The contour of integration:

- is planar
- is perpendicular to the current sheet
- is symmetrically placed with respect to the sheet
- has two of its sides parallel to ***B***



We have then:

$$\oint_{closed\ curve} \vec{B}(\vec{r})\cdot d\mathbf{s} = 2|\vec{B}|l \qquad\qquad \mu_0 I = \mu_0|\vec{K}|l,$$

where ***K*** is a surface current density (parallel to the boundary at every point).

Finally we get:

$$|\vec{B}| = \frac{1}{2}\mu_0|\vec{K}|$$

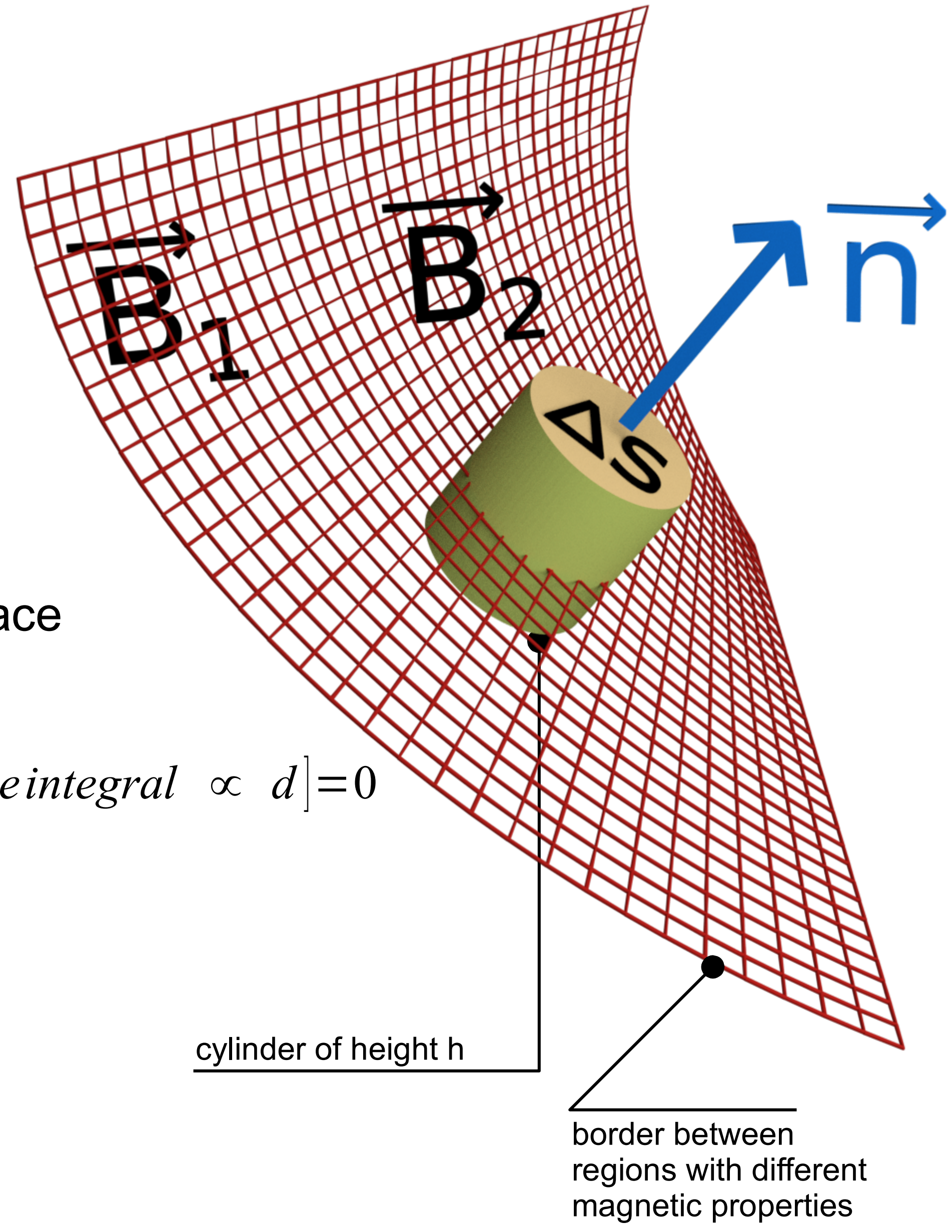
and, since ***B*** is symmetric with respect to current, for discontinuity we obtain:

$$\boxed{\Delta B = \mu_0|\vec{K}|}$$

or vectorially: $\vec{n}_2\times(\vec{B}_2-\vec{B}_1)=\mu_0\vec{K}$

this is applicable to any surface current

Continuity conditions for magnetic field



We use the fact that ***B*** is divergenceless [4]:

- the surface integral of B over the cylinder surface should be zero

$$\int_{cylinder\ surface} \vec{B}(\vec{r})\cdot d\boldsymbol{s} = (\vec{B}_2\cdot\vec{n} - \vec{B}_1\cdot\vec{n})\Delta S + [\,side\ surface\ integral \propto d\,] = 0$$

Going d→0 we get:

$$\boxed{\vec{n}\cdot(\vec{B}_2 - \vec{B}_1) = 0}$$

In magnetostatics tangential components of magnetic induction ***B*** experience discontinuity due to the presence of surface currents. The normal components of B are always continuous.

Continuity conditions for magnetic field

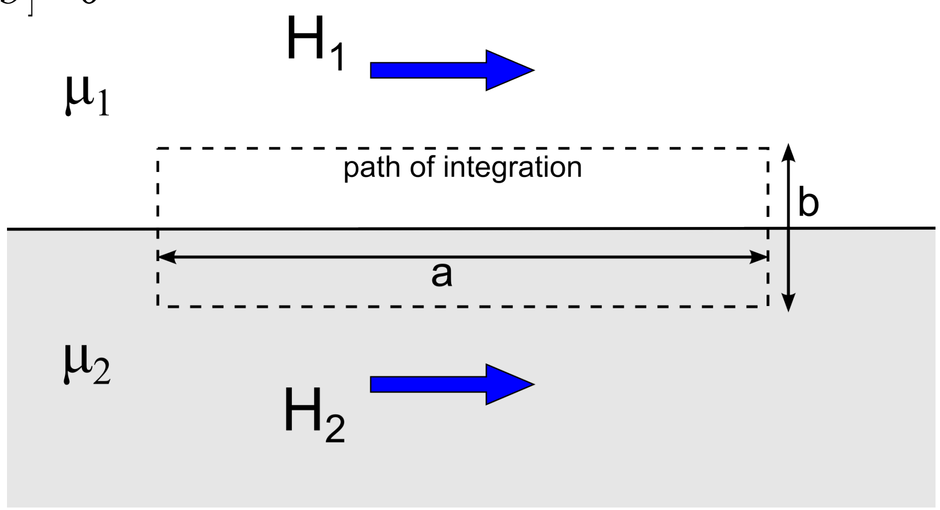
We use the fact that curl of ***H*** is null in the absence of surface currents [10]:

- the path integral of H over the rectangle edges should be zero

$$\oint_{rectangle\ edges} \vec{H}(\vec{r})\cdot d\boldsymbol{l} = (H_1 a - H_2 a) + [side\, integral \propto b] = 0$$

Going b→0 we get:

$$\vec{n}\times(\vec{H}_2-\vec{H}_1)=0$$



Continuity conditions for magnetic field

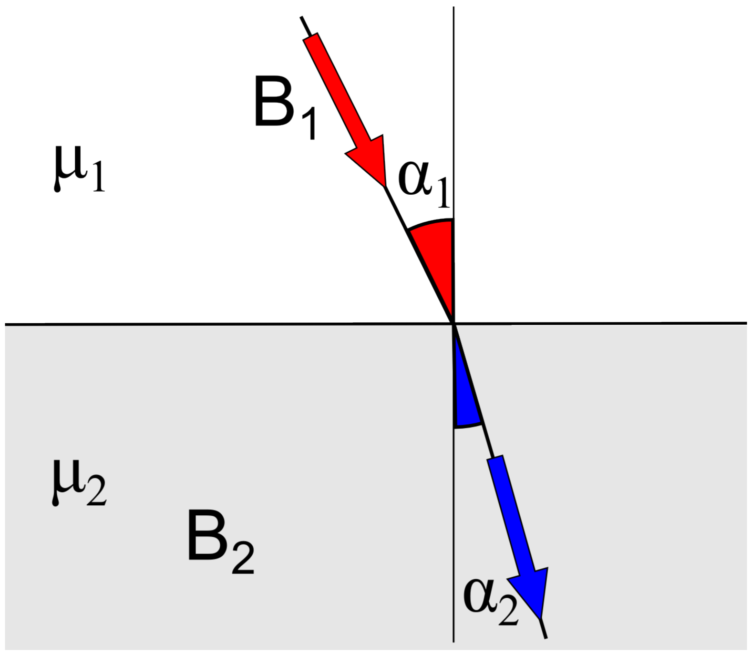
$$\vec{n}\cdot\vec{B}_2=\vec{n}\cdot\vec{B}_1 \qquad \vec{B}_2\times\vec{n}=\frac{\mu_2}{\mu_1}\vec{B}_1\times\vec{n}$$

$$\vec{n}\cdot\vec{H}_2=\frac{\mu_1}{\mu_2}\vec{n}\cdot\vec{H}_1 \qquad \vec{H}_2\times\vec{n}=\vec{H}_1\times\vec{n}$$

$$\vec{B}=\mu\,\vec{H}$$

- Law of refraction for magnetic lines [10]:

$$\frac{\tan\alpha_1}{\tan\alpha_2}=\frac{\mu_1}{\mu_2}$$



Uniqueness of solutions of magnetostatic boundary problems

We have four equations for the scalar magnetic potential ([5] A. Aharoni):

$$\nabla^2 \varphi_{inside} = \nabla \cdot \vec{M} \qquad \nabla^2 \varphi_{outside} = 0 \qquad \boxed{\varphi_{inside} = \varphi_{outside} \qquad \frac{\partial \varphi_{inside}}{\partial n} - \frac{\partial \varphi_{outside}}{\partial n} = \vec{M} \cdot \vec{n}}$$

at the boundary

Let us suppose there are two regular* functions φ_1 and φ_2 that fulfill the above equations. Then the function $\varphi_3 = \varphi_1 - \varphi_2$ must be continuous everywhere. Let us integrate:

$$\int_V (\nabla \varphi_3)^2 d^3 r = \int_V \left[\nabla \cdot (\varphi_3 \nabla \varphi_3) - \varphi_3 \underbrace{\nabla^2 \varphi_3}_{0} \right] d^3 r = \int_S \varphi_3 \frac{\partial \varphi_3}{\partial n} d\boldsymbol{S}$$

Regularity condition requires that $\frac{\partial \varphi_3}{\partial n}$ decreases as r^{-2} and φ_3 as r^{-1}. So if one extends the surface of integration to infinity (d**S** increases as r^2) the above integral vanishes. Since integrand is a square (i.e. >=0) the divergence of φ_3 vanishes. Φ_3 must be constant, but non-zero constant is not regular at infinity. $\varphi_3=0$ everywhere. We have then:

$$\boxed{\varphi_1 = \varphi_2}$$

"*There is thus only one possible solution to the potential problem of any geometry and any distribution of the magnetization. Therefore, it is never necessary to give the intermediate steps, or to justify in any other way a solution to a potential problem.*" (A. Aharoni,[5] p.111)

*A function is termed regular if and only if it is analytic and single-valued throughout a region R (mathworld.wolfram.com).

Uniqueness of solutions of magnetostatic boundary problems

“*There is thus only one possible solution to the potential problem of any geometry and any distribution of the magnetization. Therefore, it is never necessary to give the intermediate steps, or to justify in any other way a solution to a potential problem.*” (A. Aharoni,[5] p.111)

“*It should be noted, however, that while a magnetization distribution determines a unique field outside the ferromagnet, the reverse is not true. A measurement of the field outside a ferromagnetic body **is not sufficient** to determine a unique magnetization distribution that creates this field*” (A. Aharoni,[5] p.112)

Boundary conditions – an example from Jackson

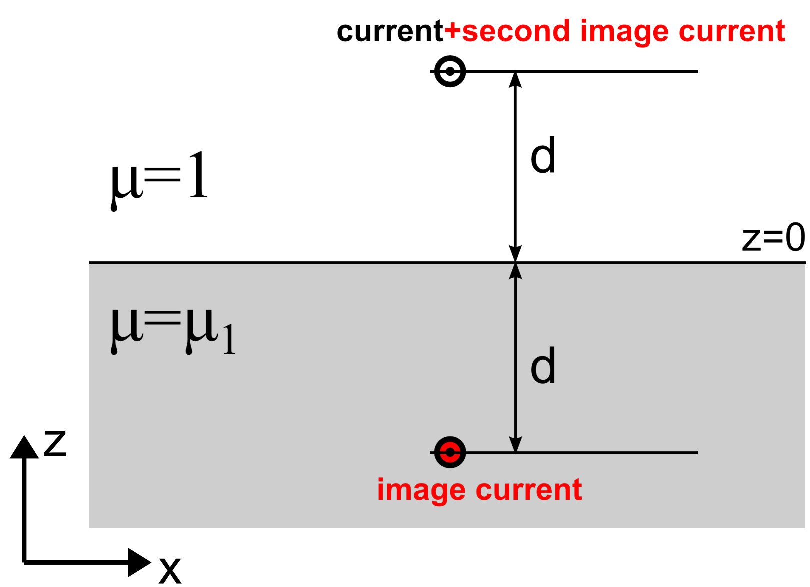
- Current line parallel to the planar boundary (z=0) separating two regions of different permeabilities
- For z>0 the the permeability is one and for z<0 it is equal to μ
- The current density is present in z>0 region
- Assume that the current flows in a line with (x=0, z) coordinates and that we are interested in the field at the boundary. We postulate that the magnetic induction can be calculated as a **superposition of the real current and two image currents**.
- For the field at (x1,y1,z1) of the current element (Jx,Jy,Jz) placed at (x,y,z) we have (Biot-Savart law): $\vec{r}_x = x1 - x$

$$\vec{B} \propto \frac{\vec{I}\times\vec{r}}{|\vec{r}|^3} = \left(\frac{Jz\,y - Jz\,y1 - Jy\,z + Jy\,z1}{((x1-x)^{2,}\,(y1-y)^{2,}\,(z1-z)^2)^{3/2}},\ \frac{-Jz\,x + Jz\,x1 + Jx\,z - Jx\,z1}{((x1-x)^2 + (y1-y)^2 + (z1-z)^2)^{3/2}},\ \frac{Jy\,x - Jy\,x1 - Jx\,y + Jx\,y1}{((x1-x)^2 + (y1-y)^2 + (z1-z)^2)^{3/2}} \right)$$

- In our case (y=0, y1=0, Jx=0, Jz=0, z1=0, x=0) this simplifies to:

$$\vec{B} \propto \frac{\vec{I}\times\vec{r}}{|\vec{r}|^3} = \left(\frac{-Jy\,z}{(x1^2 + z^2)^{3/2}},\ 0,\ \frac{-Jy\,x1}{(x1^2 + z^2)^{3/2}} \right)$$

- We assume that the image currents flow in the same direction as the existent current.
- We assume that the first image current is *mn1* times the real current and the second *mn2* times.



Boundary conditions – an example from Jackson

- For z>0 the field is the superposition of fields from the existent current and the first image current:

$$\vec{B}_{z>0}=\left(\frac{-Jy\,z+Jy\,mn1\;z}{\left(x1^2+z^2\right)^{3/2}},\;0,\;\frac{-Jy\,x1-Jy\,mn1\,x1}{\left(x1^2+z^2\right)^{3/2}}\right)$$

- For z<0 we guess that the field originates from the second image current:

$$\vec{B}_{z<0}=mn2\left(\frac{-Jy\,z}{\left(x1^2+z^2\right)^{3/2}},\;0,\;\frac{-Jy\,x1}{\left(x1^2+z^2\right)^{3/2}}\right)$$

- From boundary conditions $\vec{n}\cdot\vec{B}_2=\vec{n}\cdot\vec{B}_1$ and $\vec{B}_2\times\vec{n}=\frac{\mu_2}{\mu_1}\vec{B}_1\times\vec{n}$ we have:

$$\left(\begin{array}{l} B_z^{z<0}=B_z^{z>0} \\ B_x^{z<0}=\mu_1 B_x^{z>0} \end{array}\right)_{z1=0}$$

solving a set of equations* we get

$$mn1=\frac{\mu_1-1}{\mu_1+1}$$

$$mn2=\frac{2\,\mu_1}{\mu_1+1}$$

Multiplying the image currents by the appropriate multipliers ensures the fulfillment of magnetic boundary conditions

- Because of the uniqueness of the solutions of the magnetostatic boundary problems this is the only solution.

*Solve[{(-Jy x1-Jy mn1 x1)+Jy mn2 x1==0,(-Jy z+Jy mn1 z)/(-Jy mn2 z)-(1/mi)==0},{mn1,mn2}]

Boundary conditions – an example from Jackson

- Current line parallel to the planar boundary separating two regions of different permeability
- For z>0 the the permeability is one and for z<0 it is equal to μ.
- The current density is present in z>0 region.
- For **general current distribution** it can be shown [1] that the magnetic boundary conditions are satisfied if the following image currents are added:

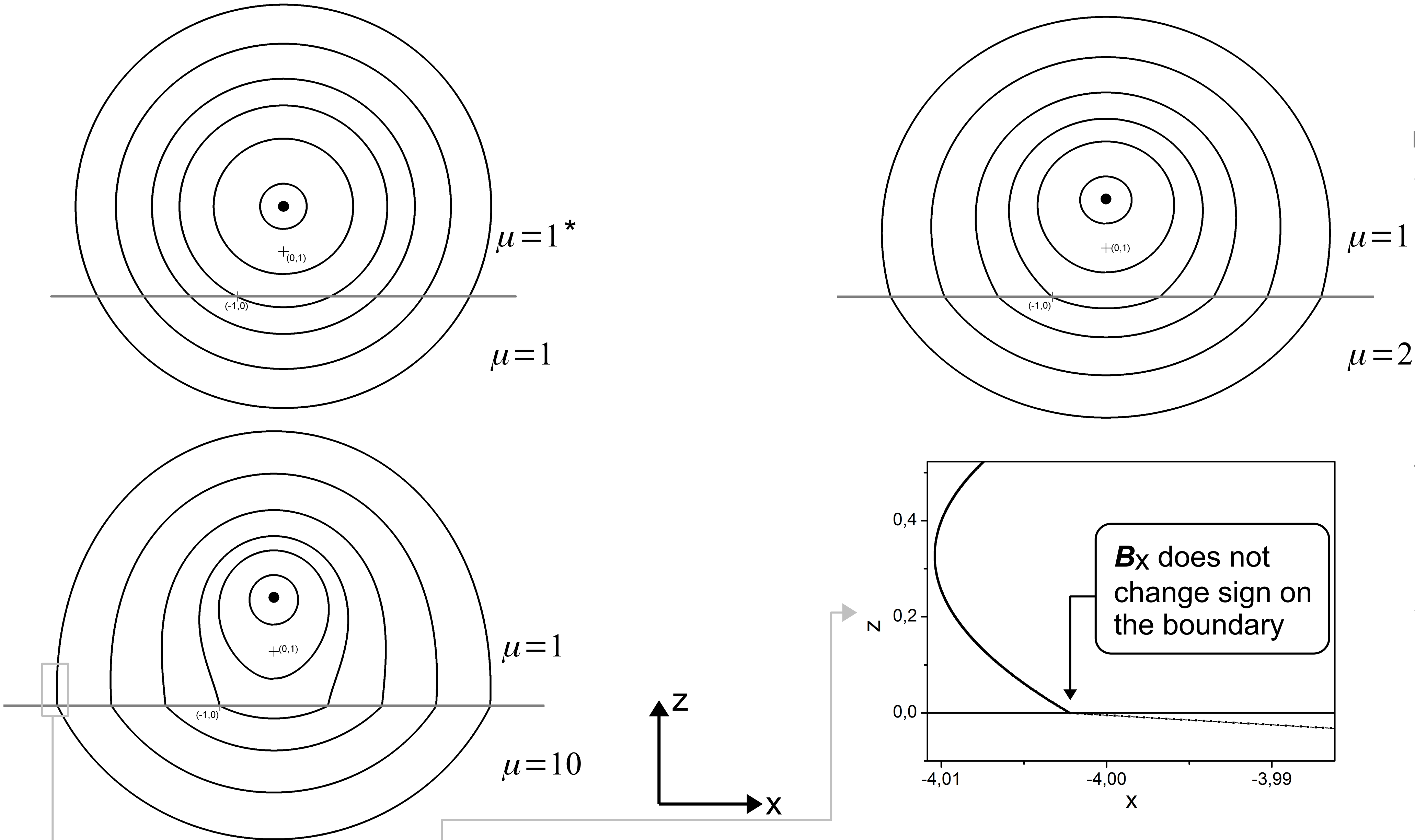
-In the region z>0 the effect of the region with permeability μ is equal to the effect of the image current **J*** of the following components (the **J*** is placed symmetrically relative to the boundary and its field is added to the field of the real current):

$$\frac{\mu-1}{\mu+1} J_x(x,y,-z), \qquad \frac{\mu-1}{\mu+1} J_y(x,y,-z), \qquad -\frac{\mu-1}{\mu+1} J_z(x,y,-z)$$

- In the region z<0 the effect of the current is that of the real current multiplied by $\frac{2\mu}{\mu+1}$
- In the region z<0 the field lines there are thus segments of circles/spheres centered at the original current line/current element

Boundary conditions – an example from Jackson

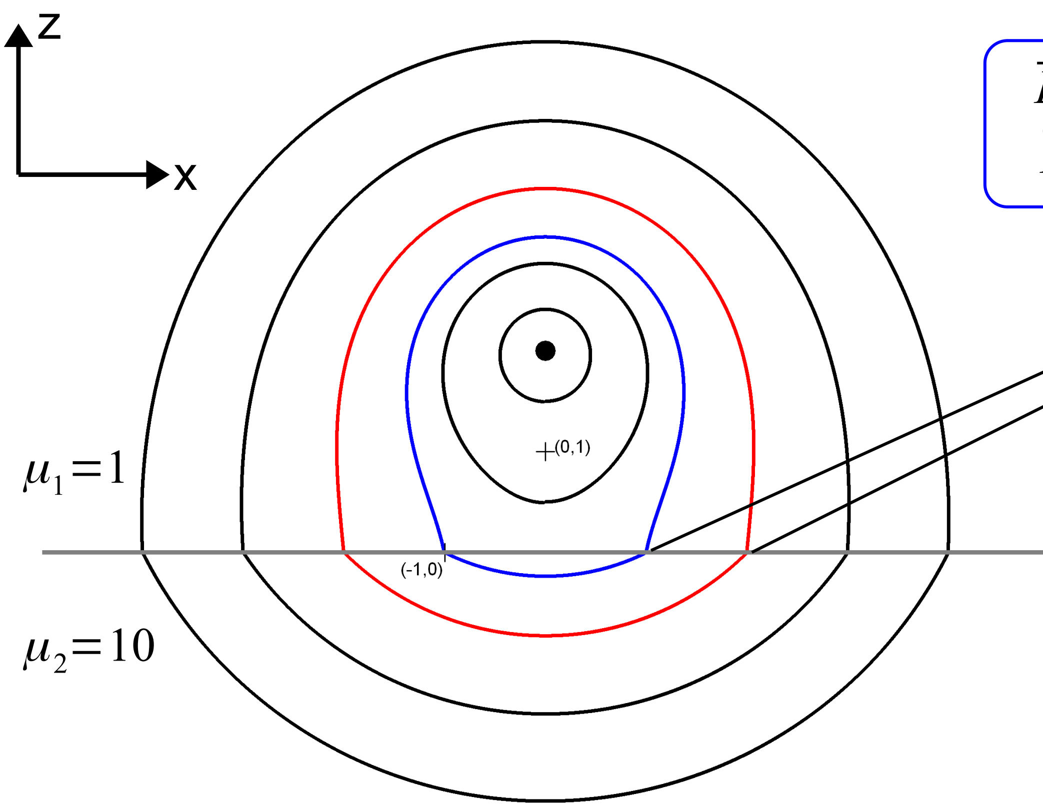
- Stream lines of ***B*** produced by a current in the vicinity of “permeability boundary”:



*the image for μ=1 in both regions is slightly scaled down (grey boundary lines in all images are from x=-5 to x=+5)

Boundary conditions – an example from Jackson

•Stream lines of ***B*** produced by a current in the vicinity of "permeability boundary":



$$\vec{B}_{\mu=1}(1{,}0)=(0.0115749\,,\ 0.0578745)$$
$$\vec{B}_{\mu=10}(1{,}0)=(0.115749\,,\ 0.0578745)$$

$$\vec{B}_{\mu=1}(2{,}0)=(0.0072343\,,\ 0.072343)$$
$$\vec{B}_{\mu=1}(2{,}0)=(0.072343\,,\ 0.072343)$$

$$\vec{n}_2\cdot(\vec{B}_2-\vec{B}_1)=0$$
$$\vec{B}_2\times\vec{n}=\frac{\mu_2}{\mu_1}\vec{B}_1\times\vec{n}$$

$$\frac{B^z_{\mu=10}}{B^z_{\mu=1}}=1$$

$$\frac{B^x_{\mu=10}}{B^x_{\mu=1}}=10$$

Magnetic field of two current sheets

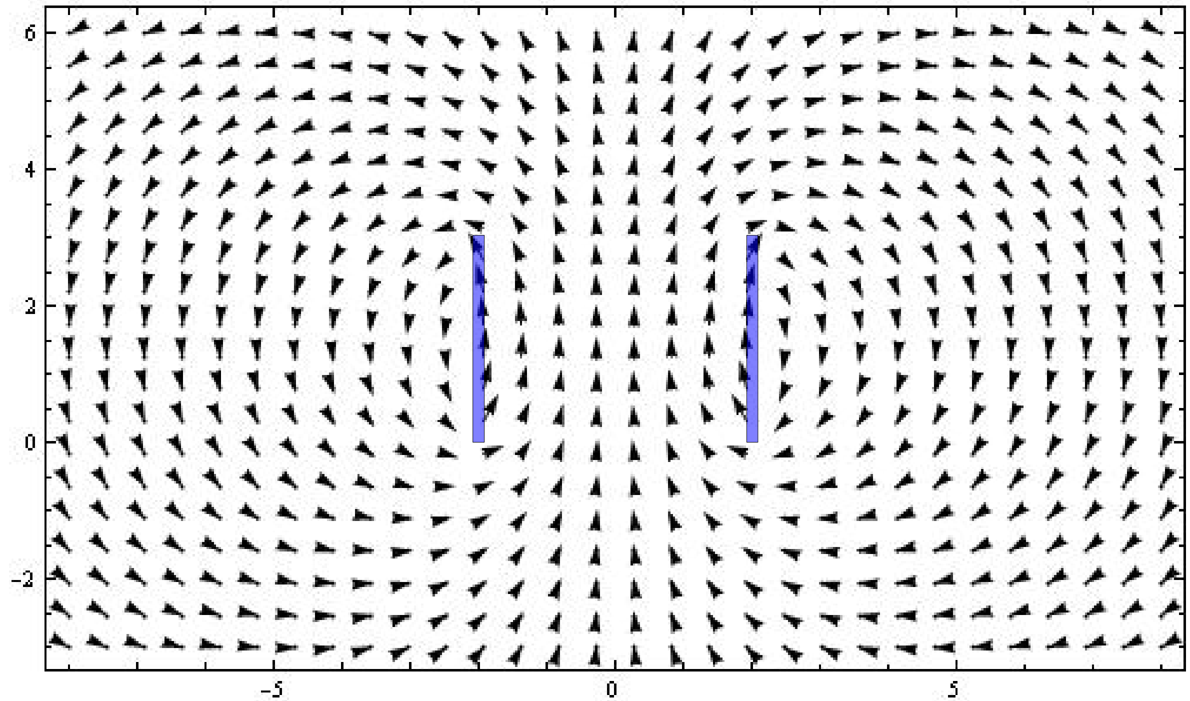
We use the same expressions for the induction of magnetic field of current sheet as previously:

$$B_z = \frac{\mu_0 K}{2\pi}\left[\arctan\left(\frac{z_2 - z}{x_2 - x_1}\right)\right]_{z0}^{z3}$$

magnetization $\vec{M}_s \Leftrightarrow \vec{K}$ surface current

$$B_x = \frac{\mu_0 K}{2\pi}\left[\ln\left[(x_2 - x_1)^2 - (z_2 - z)^2\right]\right]_{z0}^{z3}$$

, but this time for two shifted sheets with the opposite current flow direction.



x1=-2, z1=0, z3=3
x1=2, z1=0, z3=3

Magnetic field of two current sheets

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$$B_z = \frac{\mu_0 M_S}{2\pi} \left[\arctan\left(\frac{z_2 - z}{x_2 - x_1} \right) \right]_{z0}^{z3}$$

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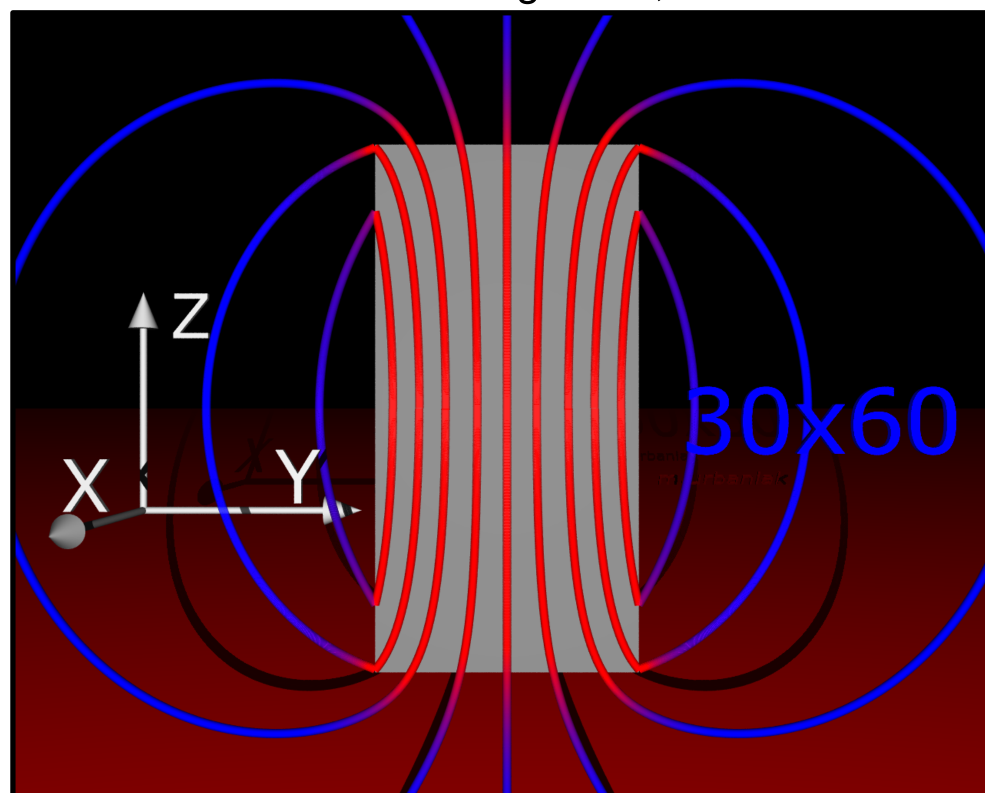
, but this time for two shifted sheets with the opposite current flow direction.

Magnetic induction ***B*** produced by two infinite current sheets corresponds to the field of infinite permanent magnet

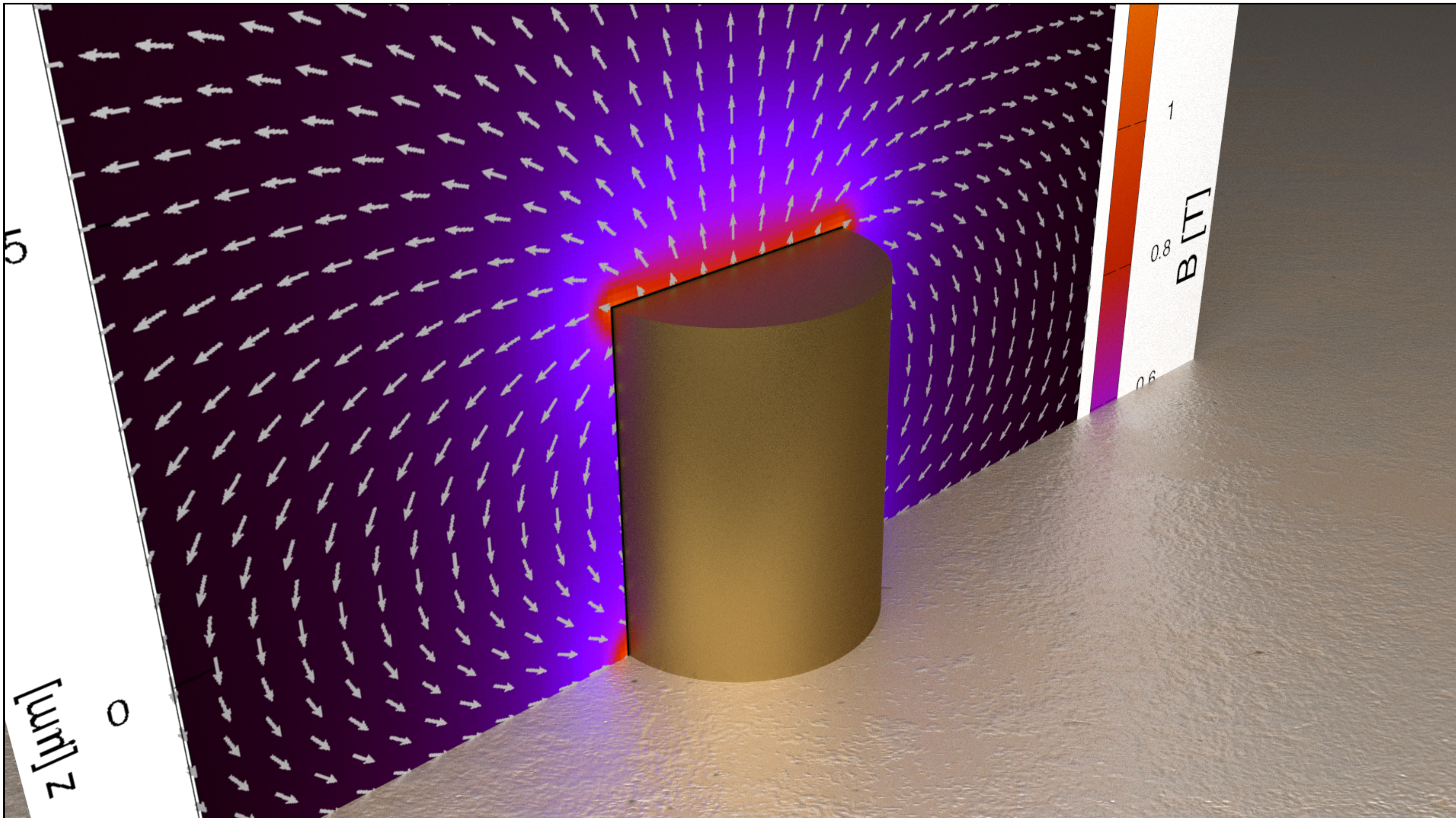
red – high field, blue – weak field



Magnetic field of thin magnets

- magnetic field of a thick cylinder

magnetic moments of the magnet point upward



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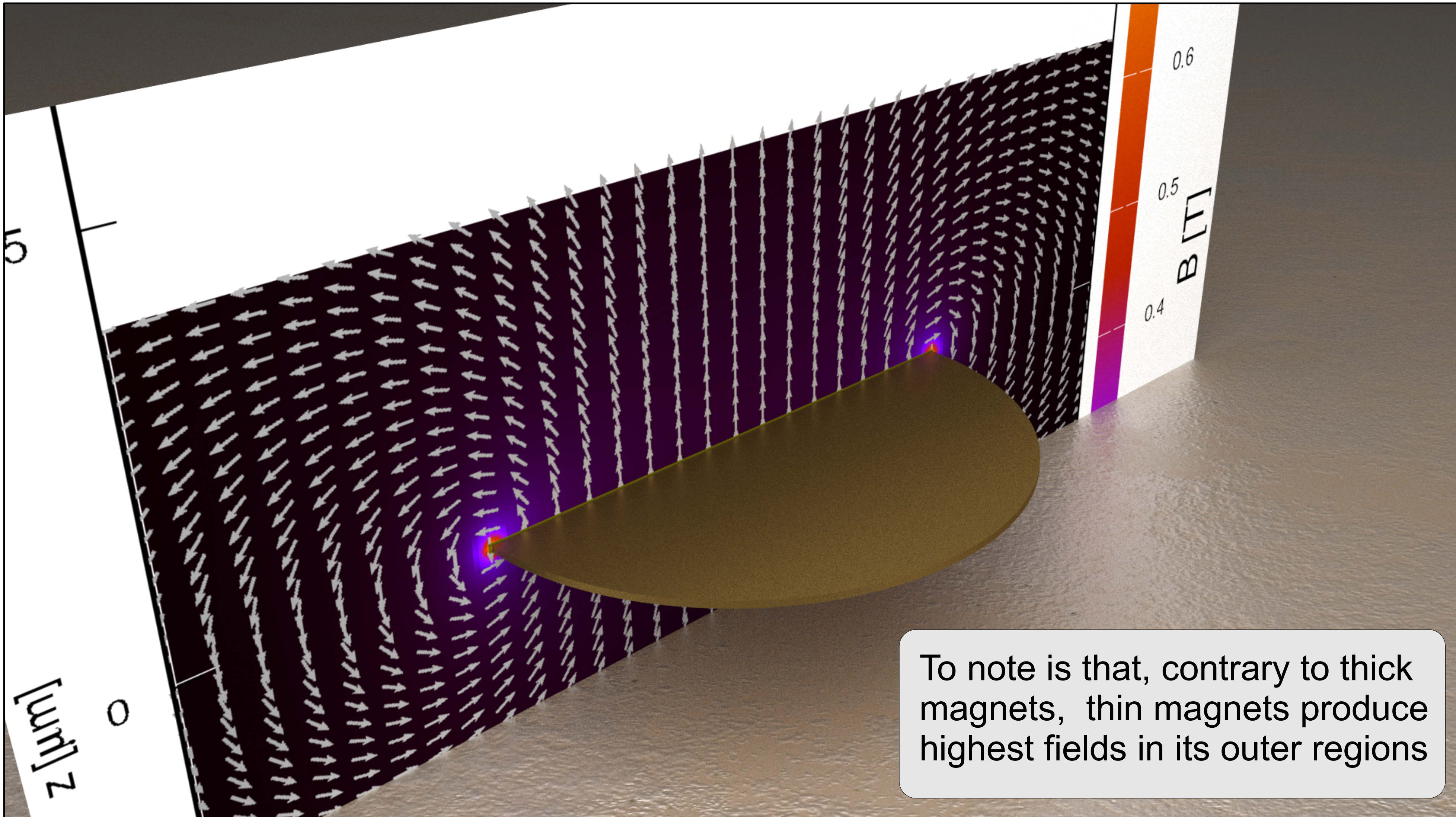
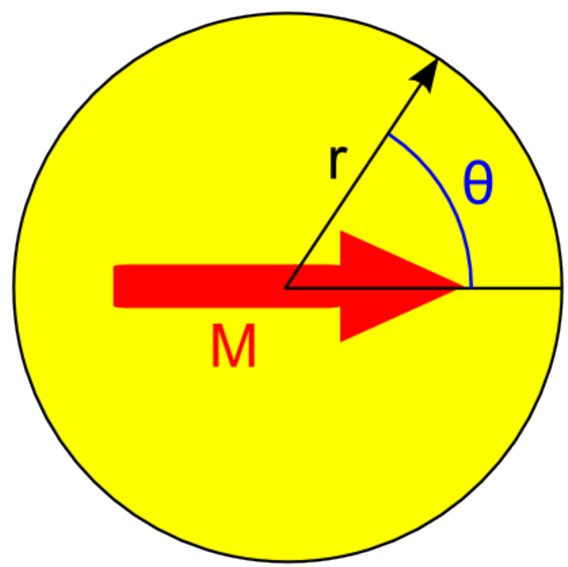
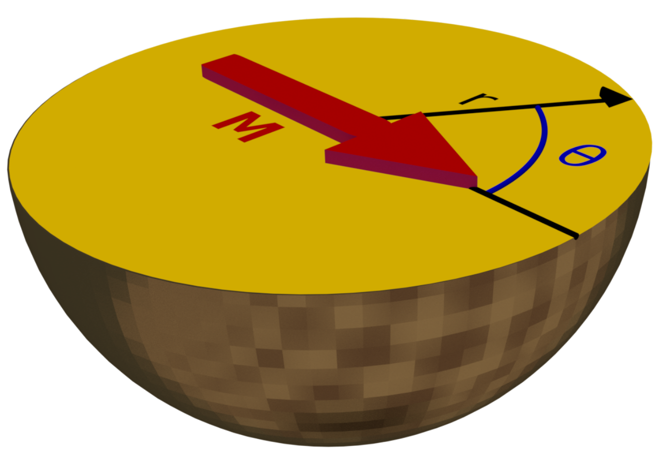


Magnetic field of uniformly magnetized sphere - example

We assume that the sphere is homogeneously magnetized in *z* direction and copying from Bartelmann [7] and Hauser [3] we seek the magnetic potential.
We have: $\vec{M}=M_0\hat{z}$

The problem is axially symmetric: we use a spherical coordinates (r, θ, φ) . The potential does not depend on azimuthal angle φ.





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$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \frac{\alpha_l P_l(\cos(\theta))}{r^{l+1}}$$

outside of the sphere

The general solution of Laplace's equation when the potential does not depend on azimuthal angle is [3]:

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \left(\beta_l r^l + \alpha_l r^{-(l+1)}\right) P_l(\cos(\theta))$$

At the origin of coordinates and the points at infinity some of the terms of the expansion above [3, p. 189] diverge. The solutions of the r^{-l} type are thus discarded within the sphere and those proportional to positive powers of r outside of it.

We can omit l=0 terms. Inside the sphere it gives just a constant ($P_0(\cos(\theta))=1$) and outside (α coefficient) produces a potential of a monopole not appearing in the multipole expansion of the current distribution [3].

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Assuming (guessing) that **i**nside the sphere induction ***B*** is parallel to *z*-axis we have:

$$\vec{H}_i=\frac{1}{\mu_0}\vec{B}_i-\vec{M}=\left(\frac{1}{\mu_0}B_0-M_0\right)\hat{z}$$

i-inside
out-outside

The continuity conditions for ***B*** on the surface of the sphere give (normal and tangential):

$$\vec{B}_i\cdot\hat{r}=\vec{B}_{out}\cdot\hat{r} \qquad \vec{H}_i\cdot\hat{\theta}=\vec{H}_{out}\cdot\hat{\theta}$$

Since it was assumed that ***B***$_{\text{inside}}$||***z*** we obtain for r-component of **B**:

$$\vec{B}_i\cdot\hat{r}=B_0\cos(\theta)=-\mu_0\frac{\partial}{\partial r}\varphi(r,\theta)\Big|_R=\mu_0\sum_{l=0}^{\infty}\alpha_l(l+1)\frac{P_l(\cos(\theta))}{R^{l+2}}=\mu_0\alpha_1\frac{2}{R^3}P_1(\cos(\theta))=\mu_0\alpha_1\frac{2}{R^3}\cos(\theta)$$

outside of the sphere

$P_1(x)=x$

$\vec{H}=-\nabla\varphi$

surface of the sphere

Magnetic field of uniformly magnetized sphere - example

From the second continuity condition follows [7]:

$$\vec{H}_i \cdot \hat{\theta} = \left(\frac{1}{\mu_0} B_0 - M_0\right)\sin(\theta) = -\frac{1}{R}\frac{\partial}{\partial \theta}\varphi(r,\theta) = \boxed{-\sum_{l=0}^{\infty} \frac{\alpha_l}{R^{l+2}} \frac{d P_l(\cos(\theta))}{d\theta}}$$

outside of the sphere*

Because of $dP_1(\cos(\theta))/d\theta = -\sin(\theta)$ we have: $P_1(x) = x$

$$\left(\frac{1}{\mu_0} B_0 - M_0\right)\sin(\theta) = -\frac{\alpha_1}{R^3}\sin(\theta)$$

and from previous page

$$B_0 \cancel{\cos(\theta)} = \mu_0 \alpha_1 \frac{2}{R^3} \cancel{\cos(\theta)}$$

Comparing coefficients of sine and cosine we get:

$$\left(\frac{1}{\mu_0} B_0 - M_0\right) = -\frac{\alpha_1}{R^3}, \quad B_0 = \mu_0 \alpha_1 \frac{2}{R^3}$$

Solving for α_1 we get:

$$\alpha_1 = \frac{1}{3} R^3 M_0$$

The scalar magnetic potential of the sphere is then:

$$\boxed{\varphi(r,\theta) = \frac{\alpha_1 P_1(\cos(\theta))}{r^2} = \frac{1}{3} R^3 M_0 \frac{\cos(\theta)}{r^2}}$$

*because we “guessed” the direction of the induction inside we do not have to make a “full” comparison (like in [3]):

$$\frac{\partial}{\partial \theta}\varphi(r,\theta)_{inside}\Big|_R = \frac{\partial}{\partial \theta}\sum_{l=0}^{\infty}\beta_l r^l P_l(\cos(\theta))\Big|_R = \frac{\partial}{\partial \theta}\sum_{l=0}^{\infty} + \alpha_l r^{-(l+1)} P_l(\cos(\theta))\Big|_R = \frac{\partial}{\partial \theta}\varphi(r,\theta)_{outside}\Big|_R$$

i.e. , in other words, we know the potential inside from our “guess”

Magnetic field of uniformly magnetized sphere - example

Taking the gradient of scalar potential we get magnetic field strength, then induction [7]:

$$\vec{B}=-\mu_0 \nabla_{r,\theta,\varphi}\varphi(r,\theta)=\frac{2}{3}\mu_0 R^3 M_0 \frac{\cos(\theta)}{r^3}\hat{r}+\frac{1}{3}\mu_0 R^3 M_0 \frac{\sin(\theta)}{r^3}\hat{\theta}$$

Which corresponds to the dipole field.

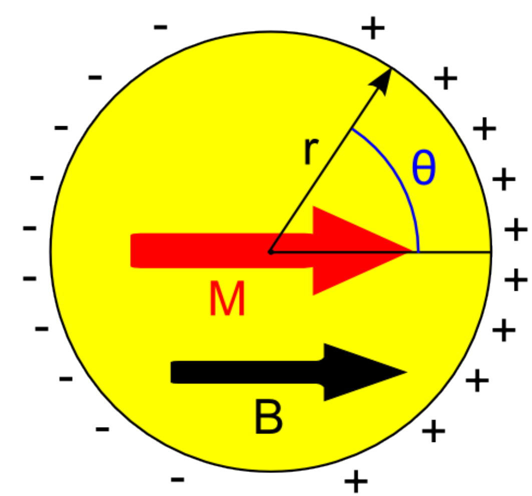
The magnetic field of uniformly magnetized sphere has dipolar character in the whole space outside the sphere.

We have a set of equations:

$$\vec{H}_i=\left(\frac{1}{\mu_0}B_0-M_0\right)\hat{z} \qquad B_0=\mu_0\alpha_1\frac{2}{R^3} \qquad \alpha_1=\frac{1}{3}R^3 M_0$$

It follows from them that:

$$\vec{H}_i=\left(\frac{1}{\mu_0}B_0-M_0\right)\hat{z}=-\frac{1}{3}M_0 \qquad \vec{B}_i=\frac{2}{3}\mu_0 M_0$$



Inside uniformly magnetized sphere magnetic induction is parallel to magnetization

Magnetizable sphere in magnetic field

$$\vec{B}=\mu_0(\vec{H}+\vec{M})=\mu_0\vec{H} \qquad \vec{M}=\chi\vec{H} \qquad (M_i=\chi_{ij}H_j)$$

volume susceptibility

usually tensor

$$\vec{B}=\mu_0(\vec{H}+\chi\vec{H})=\mu_0(1+\chi)\vec{H}=\mu_o\mu_r\vec{H}$$

relative permeability

The sphere of permeability μ is placed in an external field $\boldsymbol{B_0}$ [1,8]. The magnetization of the sphere depends now on the field strength. From previous page we have:

$$\mu_0\vec{H}_i=-\frac{1}{3}\mu_0 M$$

$$\vec{B}_i=\frac{2}{3}\mu_0 M$$

$$\xrightarrow[1]{+\vec{B}_0}$$

$$\mu_0\vec{H}_i=\vec{B}_0-\frac{1}{3}\mu_0 M$$

$$\vec{B}_i=\vec{B}_0+\frac{2}{3}\mu_0 M\equiv\mu\vec{H}_i$$

2

3

$$\mu\left(\frac{1}{\mu_0}\vec{B}_0-\frac{1}{3}\vec{M}\right)=\vec{B}_0+\frac{2}{3}\mu_0\vec{M} \qquad (1)$$

Solving (1) for ***M*** we get:

$$\vec{M}=\vec{B}_0\frac{3}{\mu_0}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)$$

4

5

$$\vec{B}_i=\vec{B}_0\left(\frac{3\mu}{\mu+2\mu_0}\right)$$

For high permeability (μ) materials:

$$\vec{B}_i=3\vec{B}_0$$ **- field amplification**

Magnetizable sphere in magnetic field

def: $\mu_0(\vec{H}+\vec{M})=\mu\vec{H}$

Since permeability is field dependent (i.e. at high fields M saturates) we have: $\mu=\mu_0+\frac{\vec{M}}{\vec{H}}$

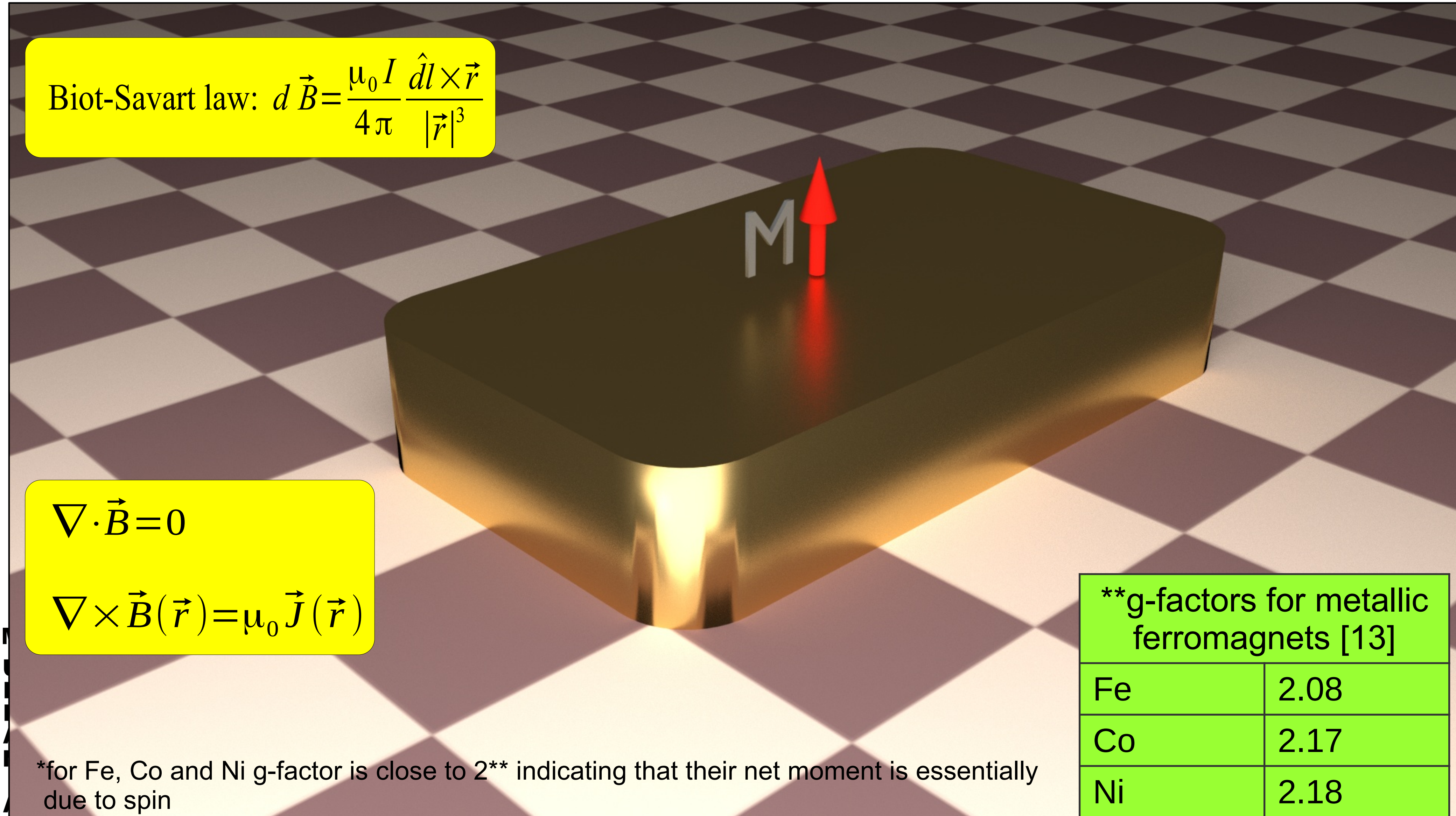
That is for high external fields: $\mu\approx\mu_0$

$\vec{B}_i=\vec{B}_0\left(\frac{3\mu}{\mu+2\mu_0}\right)\approx\vec{B}_0$ **- no field amplification**

- The amplification factor 3 is specific to a magnetizable sphere
- For other geometries (elongated rod) it can be considerably larger and is limited by the intrinsic properties of the material (magnetocrystalline anisotropy etc.)

Resume – magnetic charges, equivalent currents, scalar potential ...

Given a sample with a net magnetic moment **M** the magnetic field is produced by intrinsic, intratomic or intramolecular, currents (bound currents) and magnetic moments corresponding to spin angular momentum* of electrons, neutrons, ...



Biot-Savart law: $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\hat{dl} \times \vec{r}}{|\vec{r}|^3}$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$$

*for Fe, Co and Ni g-factor is close to 2** indicating that their net moment is essentially due to spin

**g-factors for metallic ferromagnets [13]	
Fe	2.08
Co	2.17
Ni	2.18

Resume – magnetic charges, equivalent currents, scalar potential ...

We can determine the field due to current distribution using the formula:

$$\vec{B}(\vec{r}) = \nabla \times \frac{\mu_0}{4\pi} \int \frac{J(\vec{r}\,')}{|r - r'|} d^3 r'$$

Biot-Savart law: $d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\hat{dl} \times \vec{r}}{|\vec{r}|^3}$

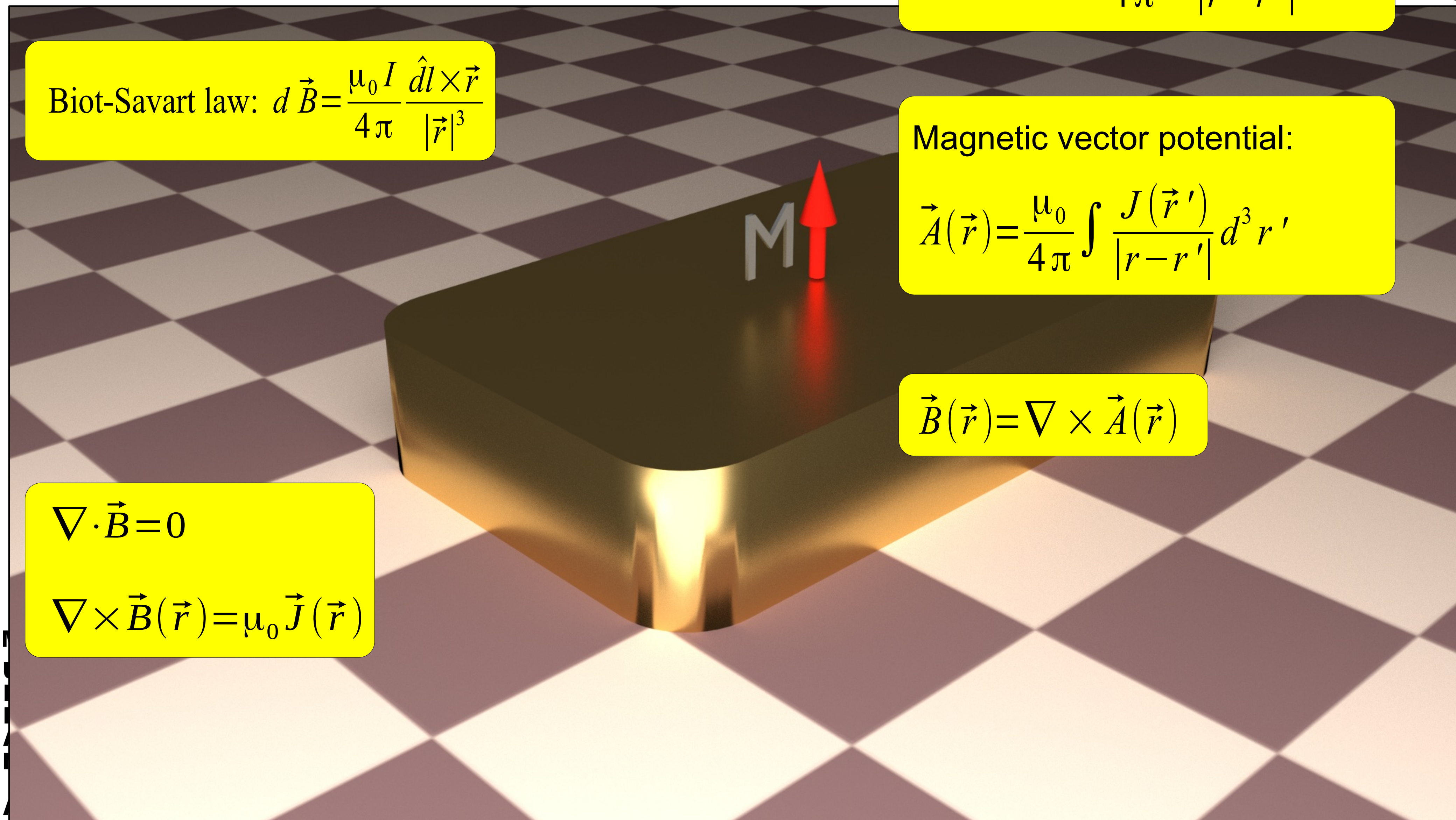
Magnetic vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{J(\vec{r}\,')}{|r - r'|} d^3 r'$$

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r})$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$$



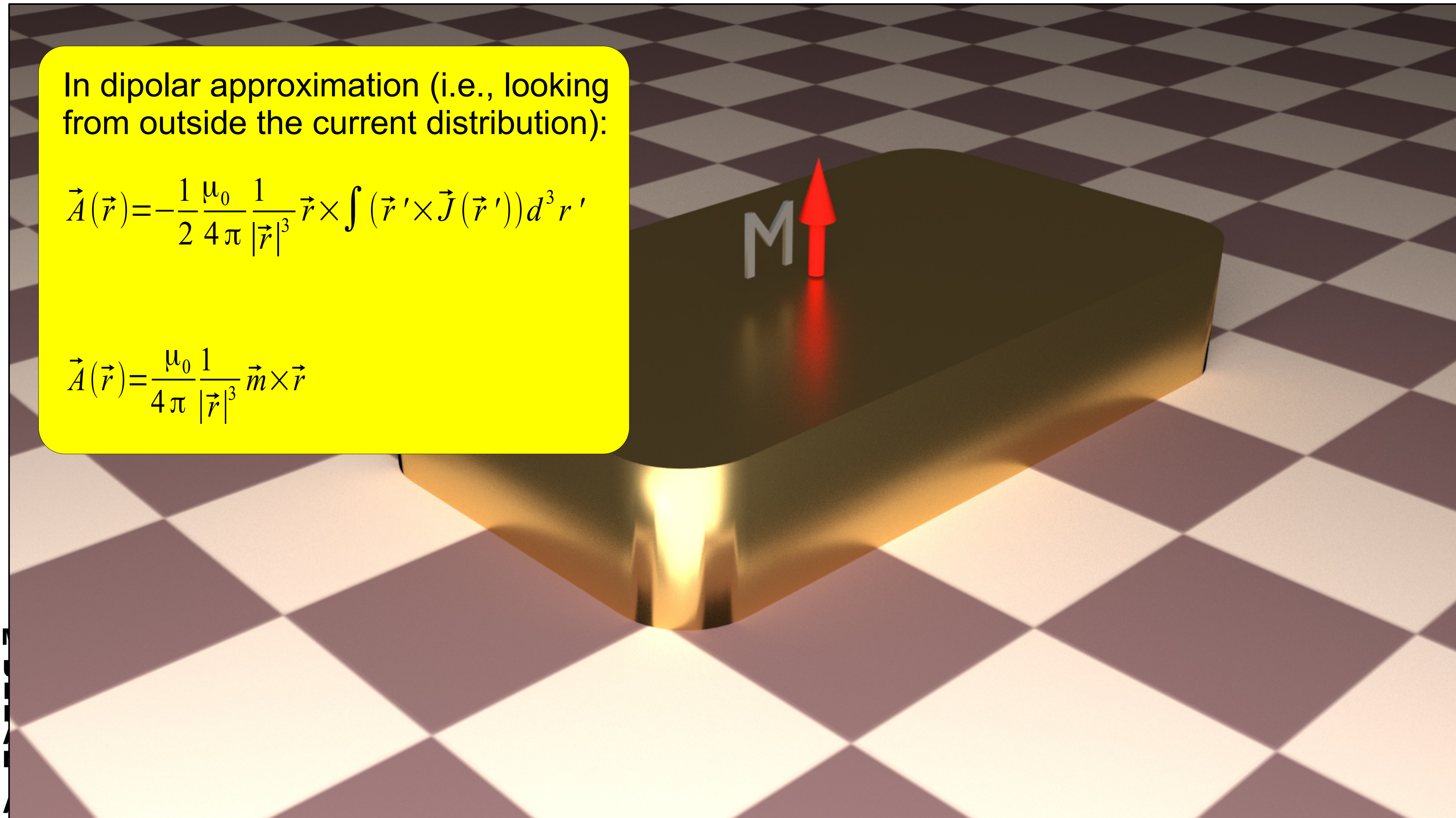
Resume – magnetic charges, equivalent currents, scalar potential ...

Magnetic moment is defined as:

$$\vec{m}=\frac{1}{2}\int\left(\vec{r}\,'\times\vec{J}(\vec{r}\,')\right)d^3r' \qquad [A\cdot m^2]$$

Its integrand is called Magnetization:

$$\vec{M}(\vec{r})=\frac{1}{2}\vec{r}\,'\times\vec{J}(\vec{r}\,') \qquad [A/m]$$



In dipolar approximation (i.e., looking from outside the current distribution):

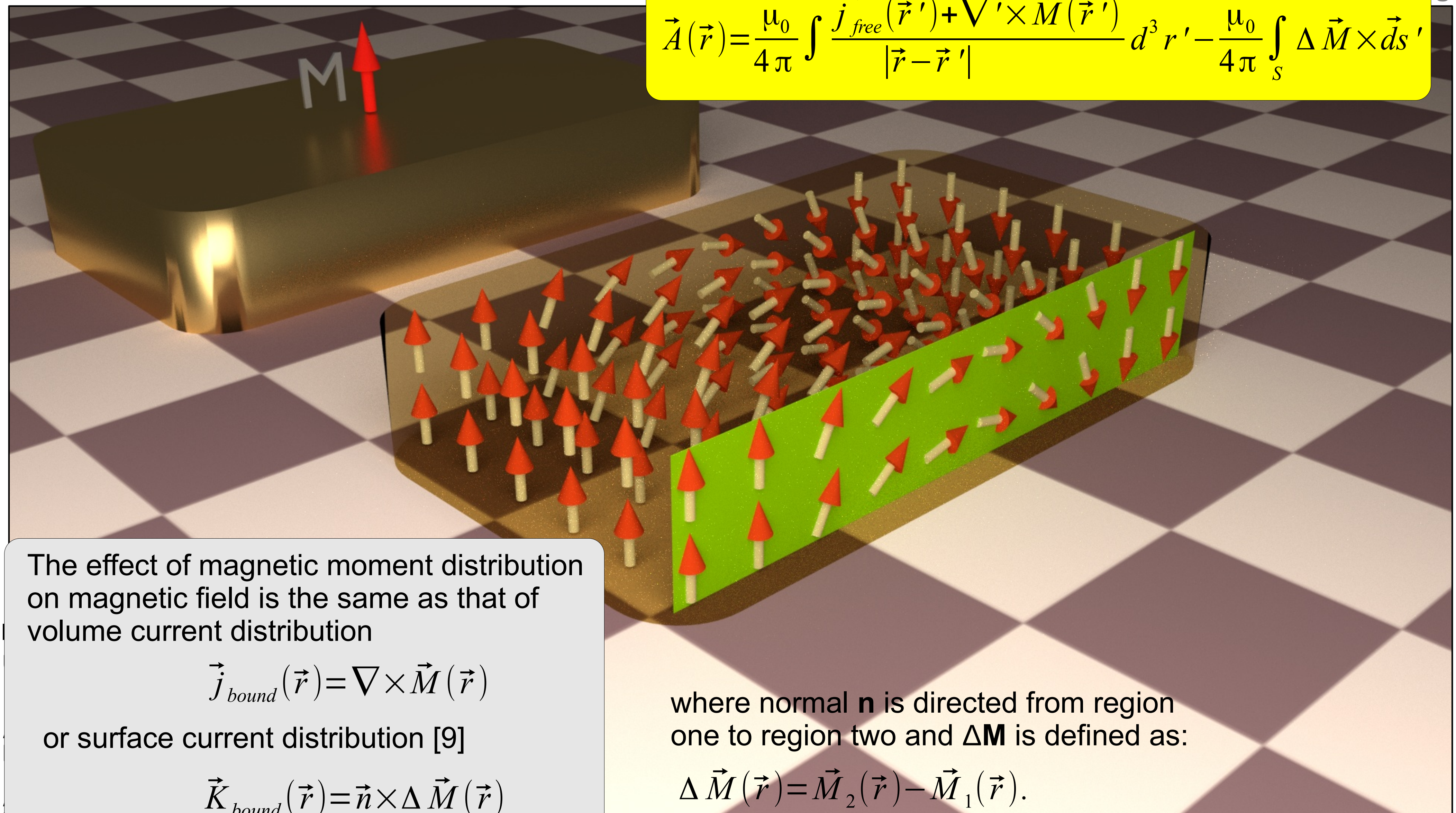
$$\vec{A}(\vec{r})=-\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|^3}\vec{r}\times\int\left(\vec{r}\,'\times\vec{J}(\vec{r}\,')\right)d^3r'$$

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|^3}\vec{m}\times\vec{r}$$

Resume – magnetic charges, equivalent currents, scalar potential ...

If we divide our sample into great many fragments, each with its small, spatially limited current distribution, we can represent them with **small magnetic moments** and integrate over the volume of the sample to obtain the the total vector potential.

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{j}_{free}(\vec{r}\,')+\nabla'\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'-\frac{\mu_0}{4\pi}\int_S \Delta\vec{M}\times\vec{ds}\,'$$



The effect of magnetic moment distribution on magnetic field is the same as that of volume current distribution

$$\vec{j}_{bound}(\vec{r})=\nabla\times\vec{M}(\vec{r})$$

or surface current distribution [9]

$$\vec{K}_{bound}(\vec{r})=\vec{n}\times\Delta\vec{M}(\vec{r})$$

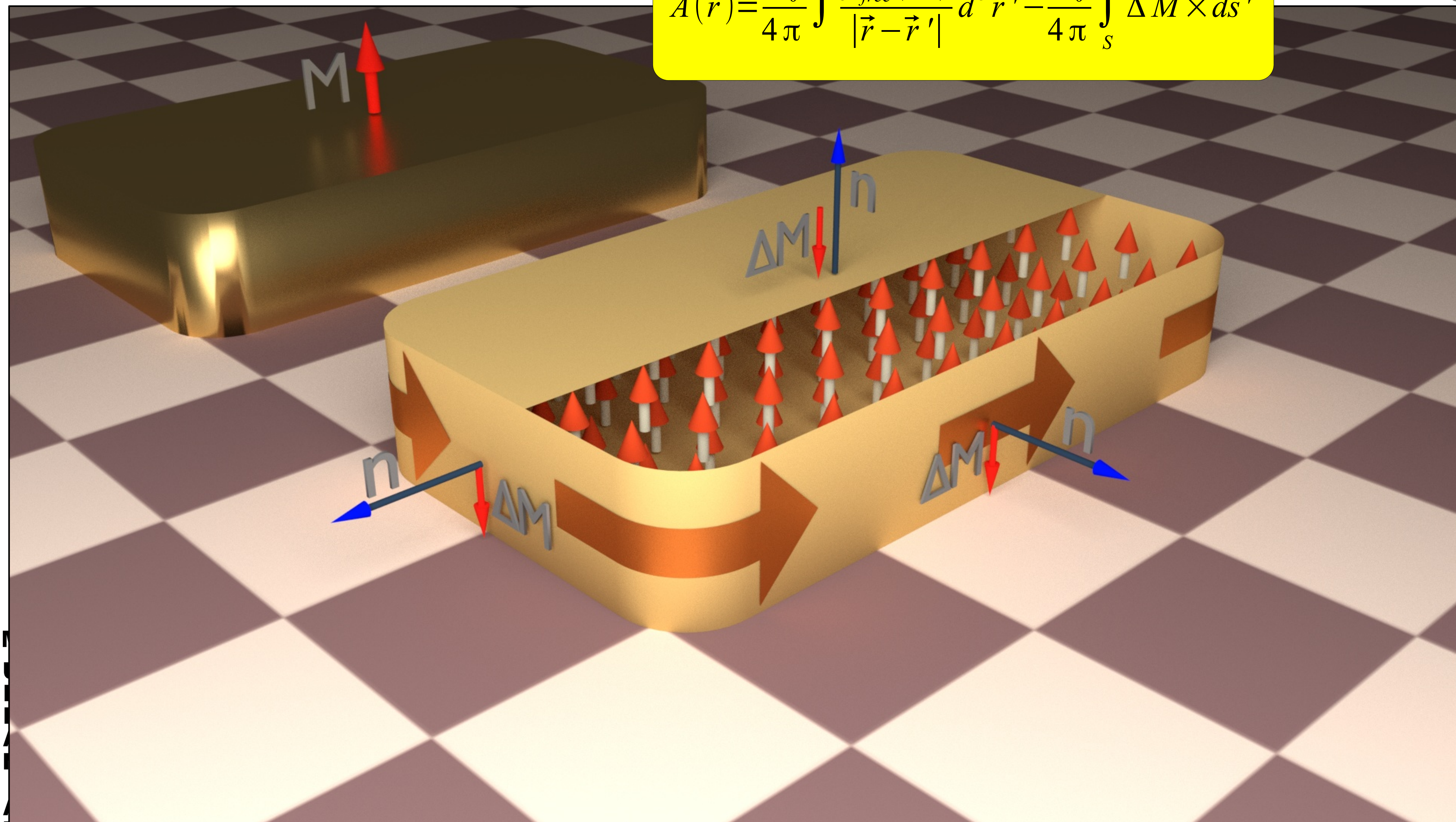
where normal **n** is directed from region one to region two and Δ**M** is defined as:

$$\Delta\vec{M}(\vec{r})=\vec{M}_2(\vec{r})-\vec{M}_1(\vec{r}).$$

Resume – magnetic charges, equivalent currents, scalar potential ...

If magnetization within the volume of the sample is **uniform** then we can omit the volume integral and calculate the field using surface integral alone:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}_{free}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3 r' - \frac{\mu_0}{4\pi}\int_S \Delta\vec{M}\times\vec{ds}\,'$$



Resume – magnetic charges, equivalent currents, scalar potential ...

If magnetization within the volume of the sample is **uniform** then we can omit the volume integral and calculate the field using surface integral alone:

The first term describes the effect of free, i.e., switchable currents

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{j}_{free}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r' - \boxed{\frac{\mu_0}{4\pi}\int_S \Delta\vec{M}\times\vec{ds}\,'}$$

We have introduced (L. 1) the magnetic field strength vector $\boxed{\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}}$

whose curl is given by $\boxed{\nabla\times\vec{H}=\vec{j}_{free}}$ and which can be expressed by a scalar magnetic

$\nabla\times\nabla\phi=0$

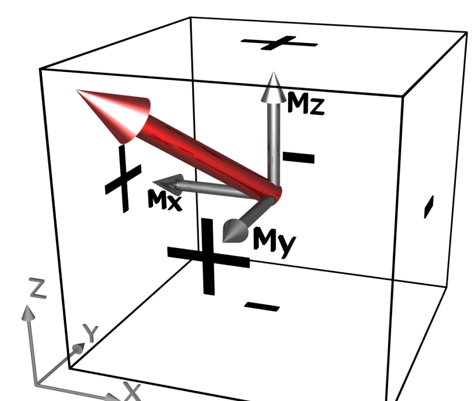
potential $\boxed{\vec{H}=-\nabla\varphi}$ provided there are no free currents in the region.

The scalar potential of the bounded dipole distribution can be thought of as originating from **fictitious** magnetic charges:

$$\phi_m(\vec{r})=\boxed{\frac{1}{4\pi}\oint_S\frac{\vec{M}\cdot\vec{ds}}{|\vec{r}|}}-\boxed{\frac{1}{4\pi}\int_V\frac{\nabla'\cdot\vec{M}}{|\vec{r}|}d^3r'}$$

$-\nabla\cdot\vec{M}$ volume magnetic charges

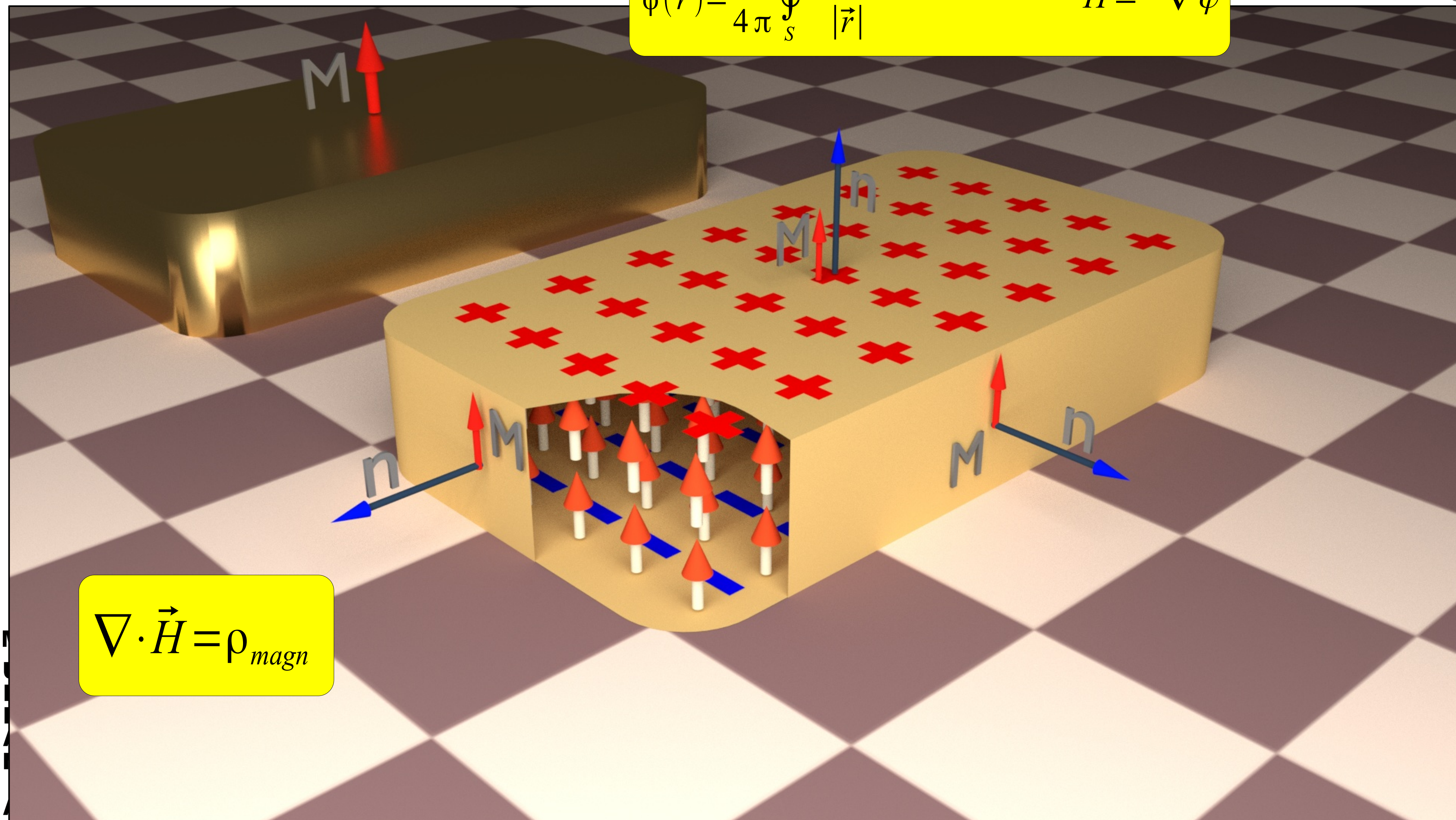
$\vec{M}\cdot\vec{ds}$ surface magnetic charges – magnetic poles



Resume – magnetic charges, equivalent currents, scalar potential ...

Similarly to the case of equivalent currents, if the magnetization within the volume of the sample is **uniform or divergenceless** then we can omit the volume integral and calculate the field using surface integral alone:

$$\phi(\vec{r}) = \frac{1}{4\pi} \oint_S \frac{\vec{M}\cdot\vec{ds}}{|\vec{r}|} \qquad \vec{H} = -\nabla\varphi$$



$$\nabla\cdot\vec{H} = \rho_{magn}$$

Resume – magnetic charges, equivalent currents, scalar potential ...

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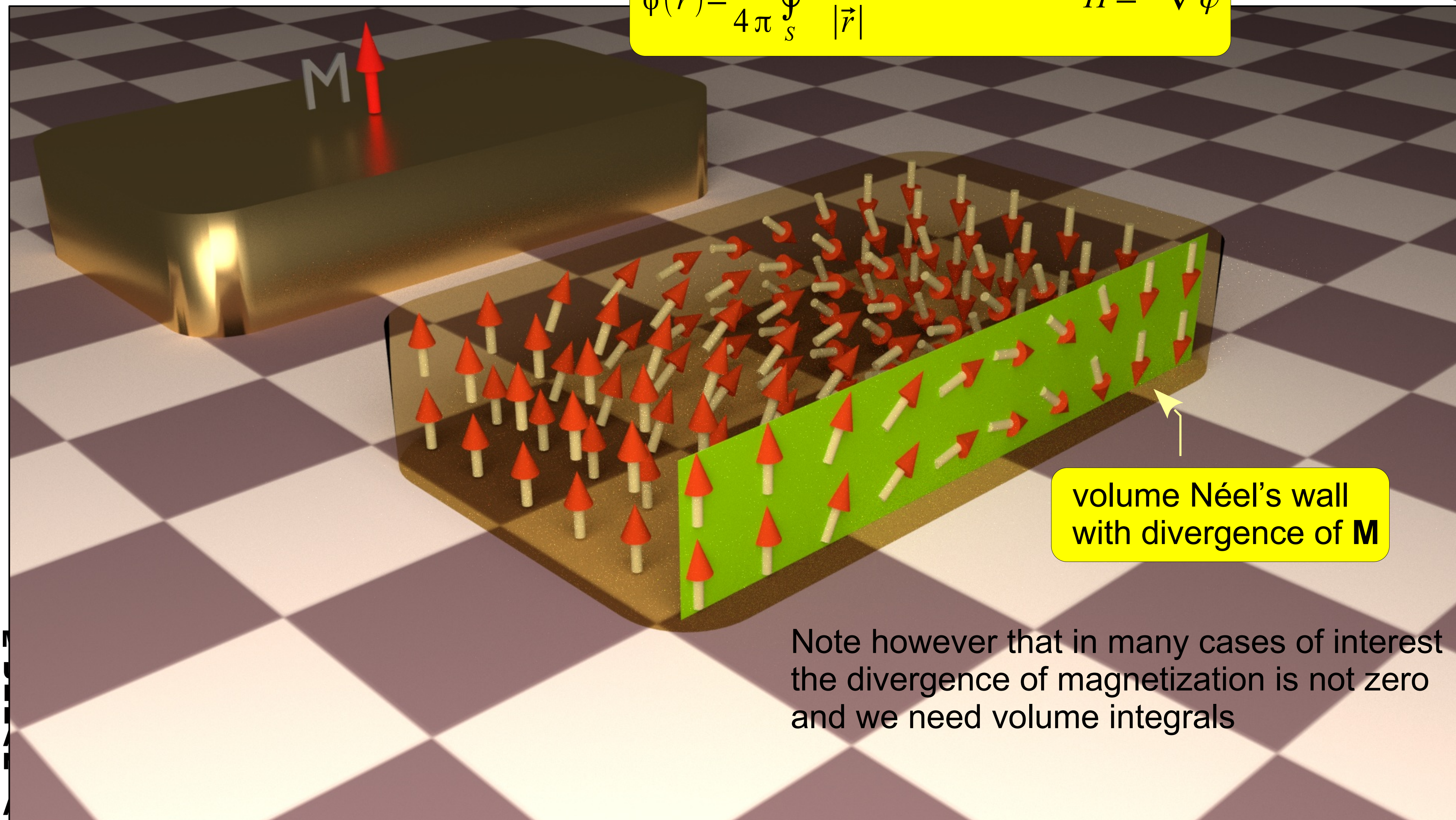
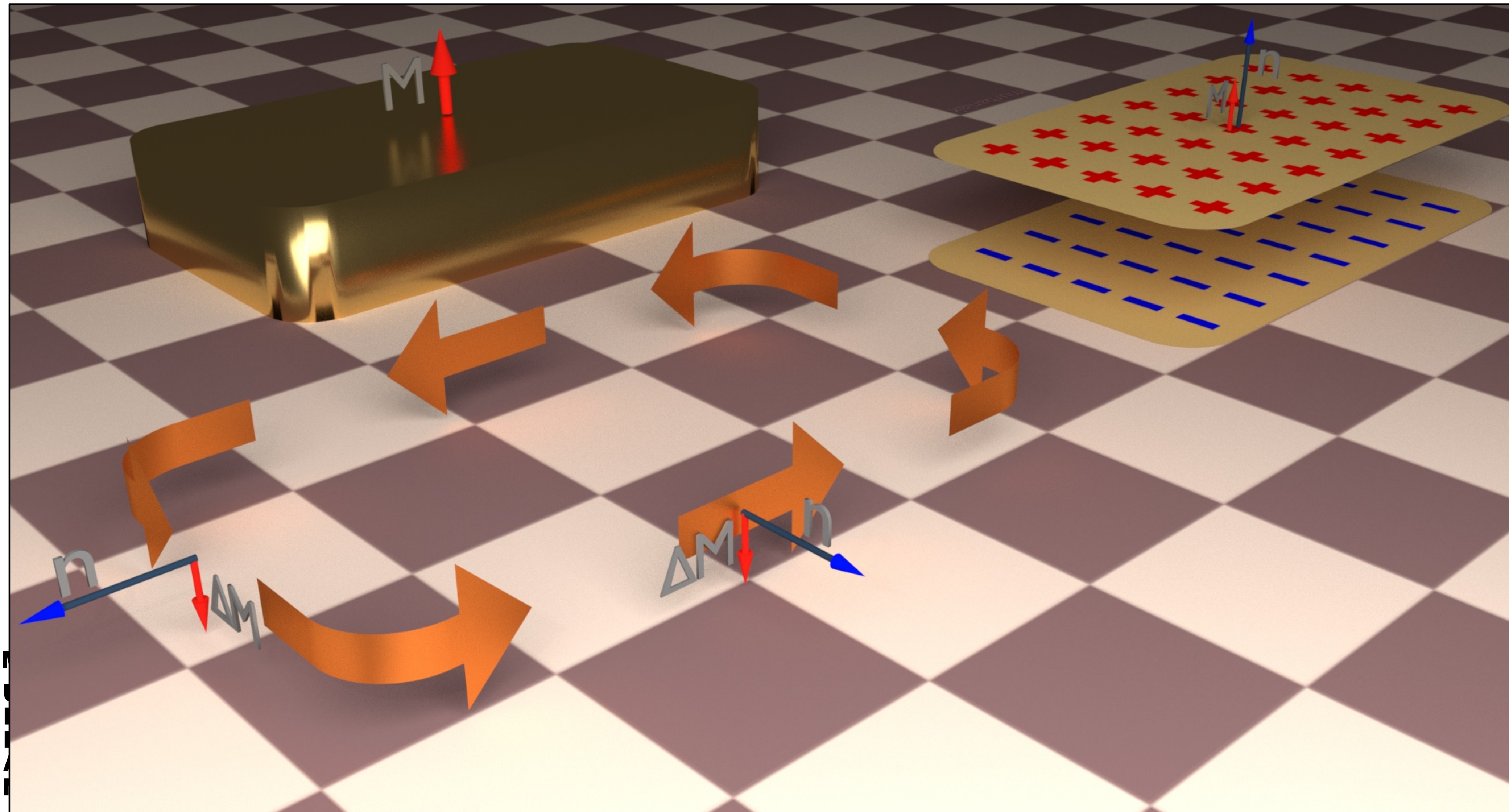


Note however that in many cases of interest the divergence of magnetization is not zero and we need volume integrals

Resume – magnetic charges, equivalent currents, scalar potential ...

Magnetic moments distribution can thus be replaced in calculations with equivalent currents (amperian approach) or equivalent fictitious magnetic charges (coulombian approach). Extensive treatment of magnetic fields calculations is given in Ref. [14]*.



* [14] H.E. Knoepfel, Magnetic fields: A comprehensive theoretical treatise for practical use, John Wiley & Sons, Inc., 2000

Potential energy of magnetic charge

In many cases instead of calculating energy of a magnetic body in the external field from the formula:

$$E_{magn}=-\int \vec{M}\cdot\vec{B}\,dV$$

one can use *magnetic charges* and *magnetic scalar potential* [15]: $\vec{H}=-\nabla\varphi$

$$E_{magn}=-\mu_0\int \vec{M}\cdot\vec{H}\,dV=\mu_0\int \vec{M}\cdot\nabla\phi\,dV=\mu_0\int(\hat{x}M_x+\hat{y}M_y+\hat{z}M_z)\cdot\left(\hat{x}\frac{\partial}{\partial x}\phi+\hat{y}\frac{\partial}{\partial y}\phi+\hat{z}\frac{\partial}{\partial z}\phi\right)dV=$$

$$\mu_0\int\left(M_x\frac{\partial}{\partial x}\phi+M_y\frac{\partial}{\partial y}\phi+M_z\frac{\partial}{\partial z}\phi\right)dV=$$

derivative of the product

$$\mu_0\int\left[\left(\frac{\partial}{\partial x}M_x\phi-\phi\frac{\partial}{\partial x}M_x\right)+\left(\frac{\partial}{\partial y}M_y\phi-\phi\frac{\partial}{\partial y}M_y\right)+\left(\frac{\partial}{\partial z}M_z\phi-\phi\frac{\partial}{\partial z}M_z\right)\right]dV=$$

$$\mu_0\int\left[\frac{\partial}{\partial x}M_x\phi+\frac{\partial}{\partial y}M_y\phi+\frac{\partial}{\partial z}M_z\phi\right]dV-\mu_0\int\left[\phi\frac{\partial}{\partial x}M_x+\phi\frac{\partial}{\partial y}M_y+\phi\frac{\partial}{\partial z}M_z\right]dV=$$

$$\mu_0\int\nabla\cdot(\phi\vec{M})\,dV-\mu_0\int\phi\nabla\cdot\vec{M}\,dV=\mu_0\int_{surface}\phi\vec{M}\,d\vec{S}-\mu_0\int\phi\nabla\cdot\vec{M}\,dV$$

Gauss divergence theorem

$$E_{magn}=\mu_0\int_{surface}\phi\,\sigma_{magn}\,ds-\mu_0\int\phi\,\rho_{magn}\,dV$$

- surface and volume integrals of the product of magnetic charge and magnetic potential
- for uniformly magnetized sample only surface integral necessary

[15] A. Hubert, W. Rave, S.L. Tomlinson, phys. stat. sol. (b) **204**, 817 (1997)

Demagnetizing factor

If and only if the surface of uniformly magnetized body is of second order the magnetic induction inside is uniform and can be written as:

$$\vec{B} = \mu_0 (-N \cdot \vec{M} + \vec{M})$$

N is called the demagnetizing tensor [5]. If magnetization is parallel to one of principle axes of the ellipsoid *N* contracts to **three numbers** called demagnetizing (or demagnetization) factors sum of which is one:

$$N_x + N_y + N_z = 1$$

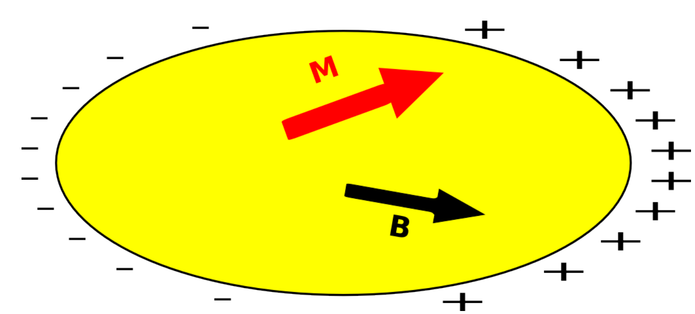
For the sphere, from symmetry considerations:

$N_x = N_y = N_z = \frac{1}{3}$ from which we obtain: $\vec{B} = \mu_0 (-\frac{1}{3} \vec{M} + \vec{M}) = \mu_0 \frac{2}{3} \vec{M}$ - see 3 slides back

For a general ellipsoid magnetization and induction are not necessarily parallel.

Demagnetization decreases the field inside ferromagnetic body.

Demagnetizing tensor describes just the influence of the body's shape on magnetic field inside it. The tensor/factor is only auxiliary quantity.



Demagnetizing factor

In some limiting cases the calculable demagnetizing factor of the ellipsoid can be used for the calculation (*approximate*) of the fields inside bodies of other shapes.

- For infinite cylinder (in *z*-direction) there is no discontinuity of magnetization along *z*-axis so:

$N_z=0$ and because of axial symmetry: $N_x=N_y=\frac{1}{2}$

In the infinite cylinder magnetized along its long axis the induction is*: $\vec{B}=\mu_0\vec{M}$

- For infinite planar sample perpendicular to *z*-axis we have no magnetic charges along *x* and *y* axes:

$N_z=1$ and because of axial symmetry: $N_x=N_y=0$

In the infinite planar sample magnetized in-plane the induction is*: $\vec{B}=\mu_0\vec{M}$

*Plus the external field if present.

Demagnetizing factor

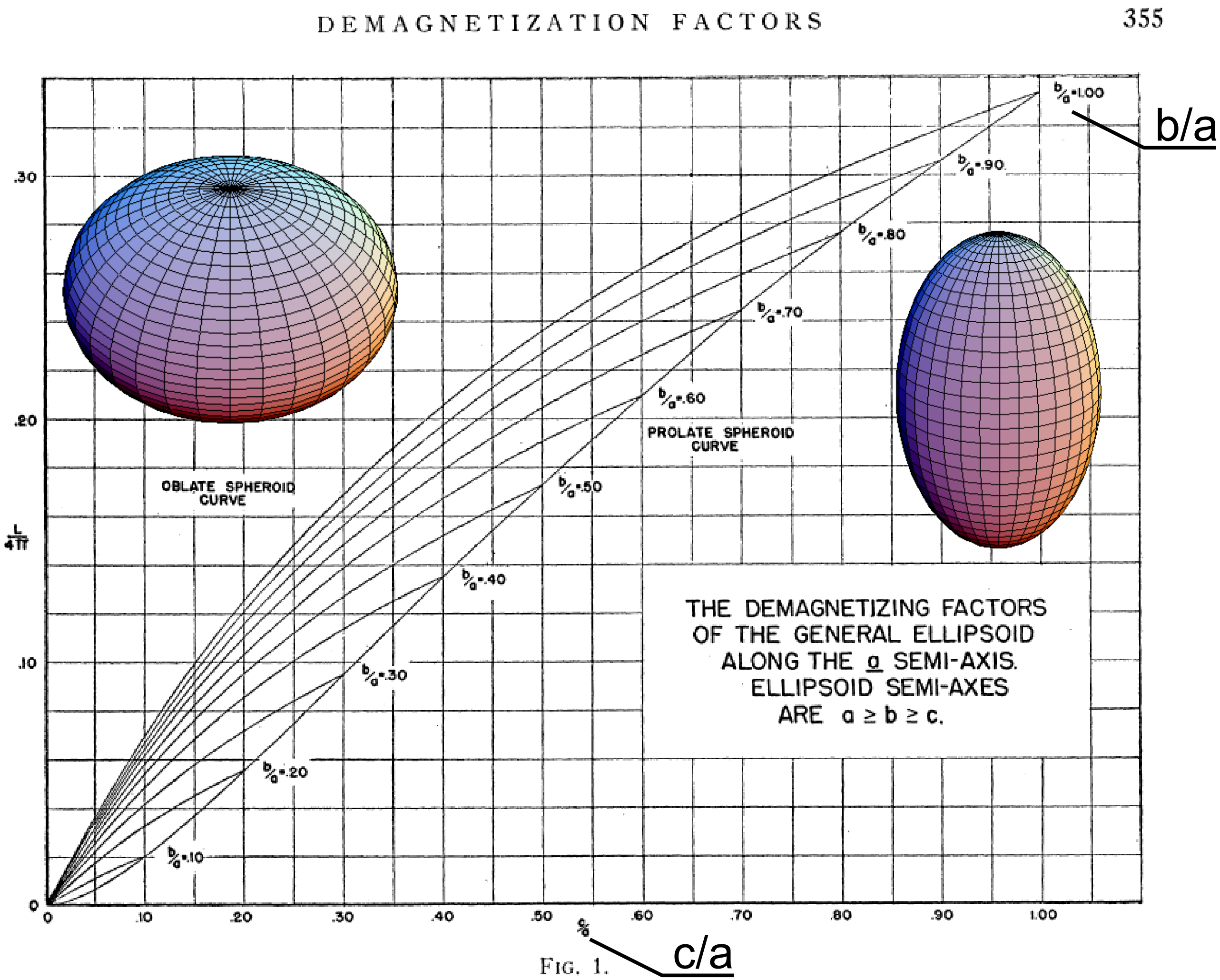


FIG. 1.

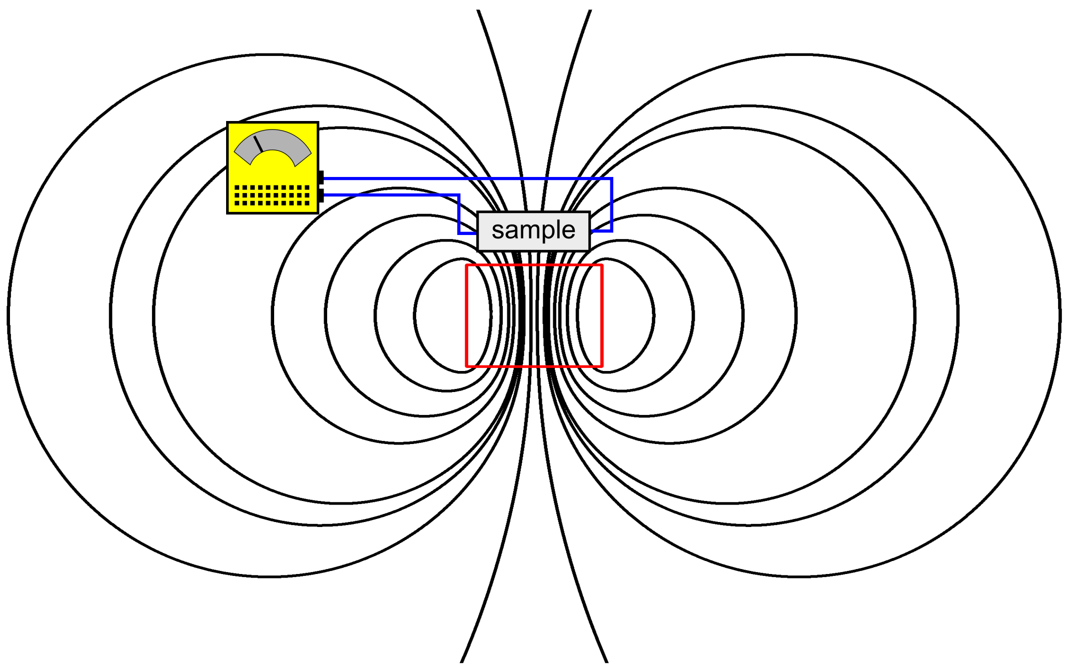
L/4π – demag. factor along the longest semi-axis (a)

J.A. Osborn, Phys.Rev. 1945, 67 (351)

images of spheroids from Wikimedia Commons; author: Cdw1952

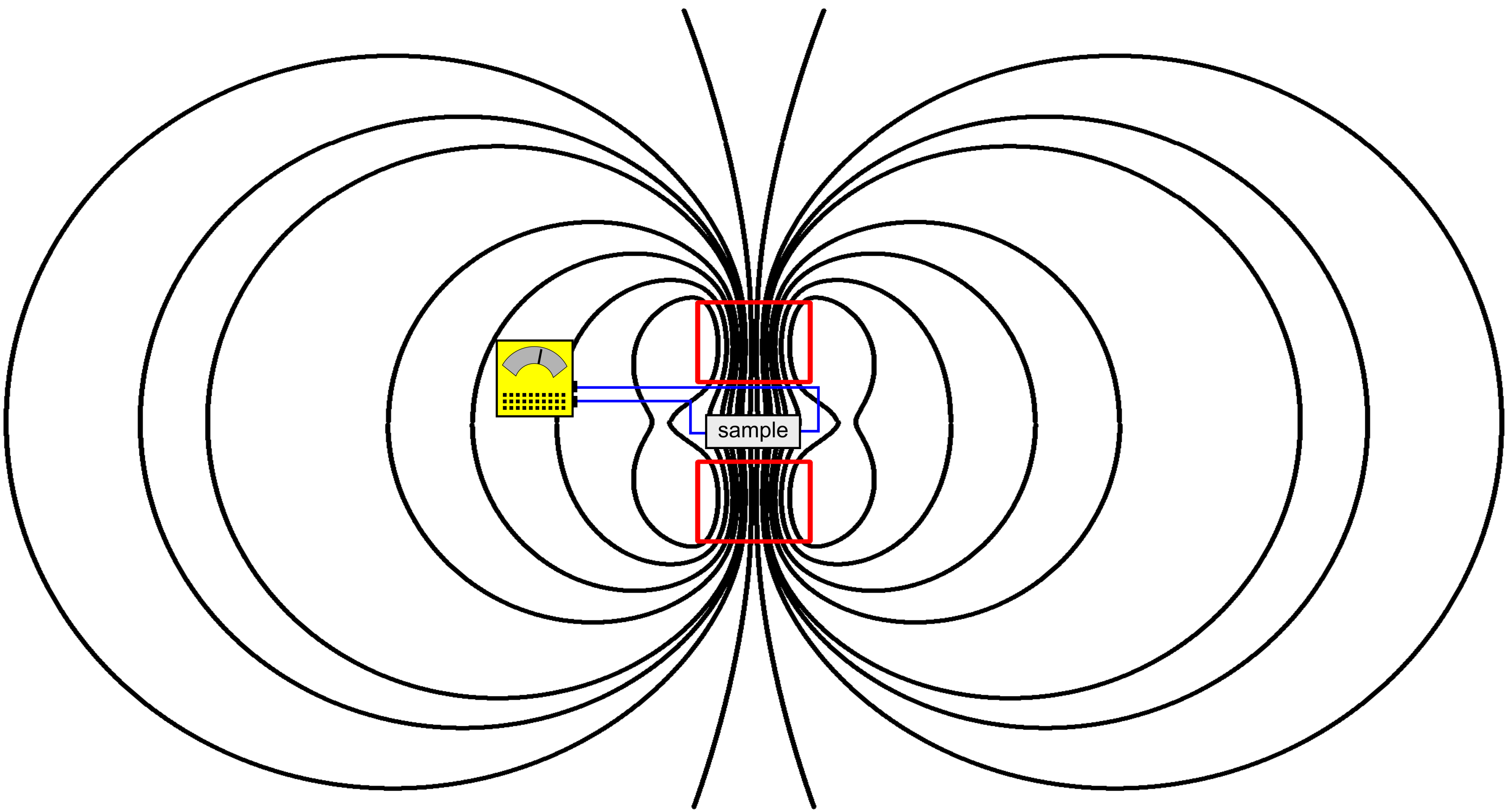
Sources of magnetic fields for measurements

- In open magnetic circuits of typical sizes the spatial variations of the intensity and direction of magnetic induction ***B*** are to high to provide enough space for experiments involving homogeneous magnetic field.
- Field of one magnet: **sample in highly inhomogeneous field**



Sources of magnetic fields for measurements

- In open magnetic circuits of typical sizes the spatial variations of the intensity and direction of magnetic induction ***B*** are to high to provide enough space for experiments involving homogeneous magnetic field.
- Field of two magnets: **sample in highly homogeneous field**, of higher strength, *if gap is narrow*



Special purpose magnets configuration - examples

from fridgemagnets-ecard.appspot.com

Refrigerator magnets (to stick things to refrigerator etc.)

- they use special configuration of magnetization to obtain **one-sided flux*** – no magnetic field is present on other side
- If the magnetization vector is constant and rotates clockwise when viewed moving from left to right:

$$M_x = M_0 \sin(kx) \qquad M_y = M_0 \cos(kx) \qquad M_z = 0$$

then the flux emerges exclusively below the structure.

- Because M_x depends on x the divergence of $\boldsymbol{M}$ is:

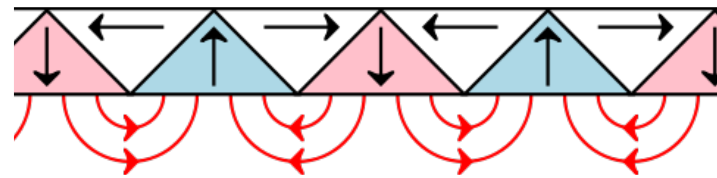


image source Wikimedia Commons; author Pieter Kuiper

$$\nabla \cdot \vec{M} = M_0 k \cos(kx) \neq 0$$ → **magnetic charges within the tape (or film)**

Within the tape the scalar potential must obey Poisson's equation (below and above Laplace's): $\nabla^2 \varphi(x, y) = M_0 k \cos(kx)$

The solution is on-sided flux: with regard to the upper region, surface and volume poles conceal each other exactly.

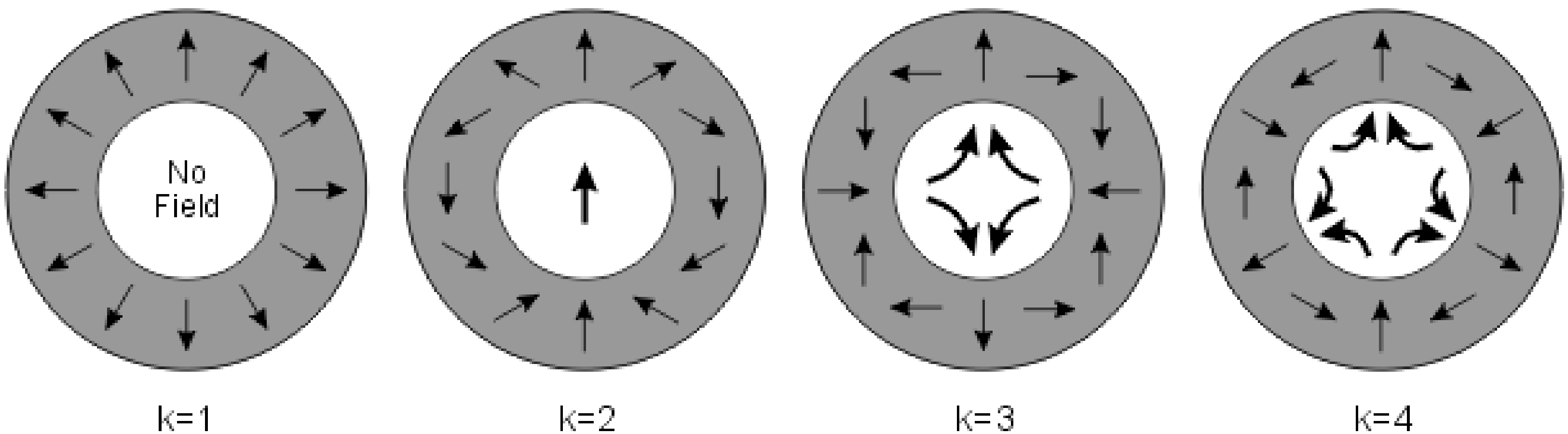
•The one-sided flux increases the holding force almost by a factor of 2.

•Note that spatially alternating magnetization increases gradient of magnetic field ***B***.

*H.A. Shute, J.C. Mallinson, D.T. Wilton, D.J. Mapps, IEEE 2000, **36** (440)

Special purpose magnets configuration - examples

Halbach cylinders:
- use the same principle as refrigerator magnets to create uniform field within spacious volume
- allow high field magnetic measurements with **very-low power consumption**: two coaxial Halbach cylinders can produce magnetic field of arbitrary direction

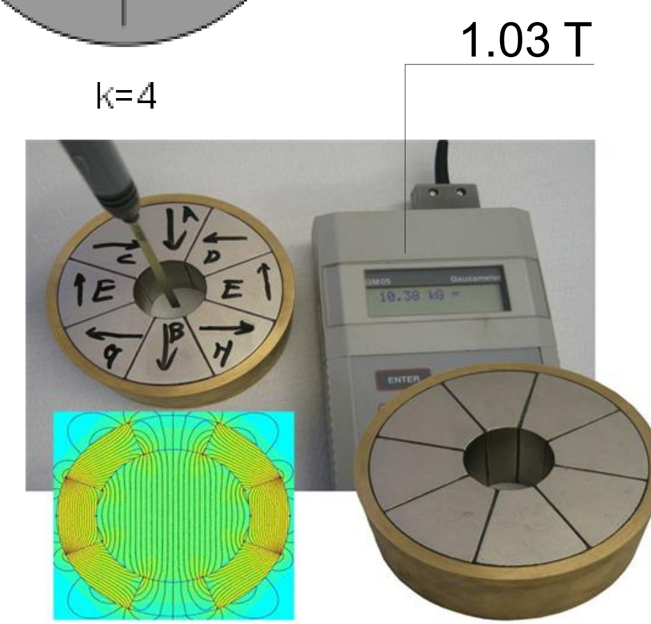


source Wikimedia Commons; author:User:Hiltonj

For k=2 the uniform field in the centre of the cylinder is*:

$$\vec{B} = \mu_0 M_0 \ln\left(\frac{r_{outer}}{r_{inner}}\right)$$

The induction may be greater than $\mu_0 M_0$.



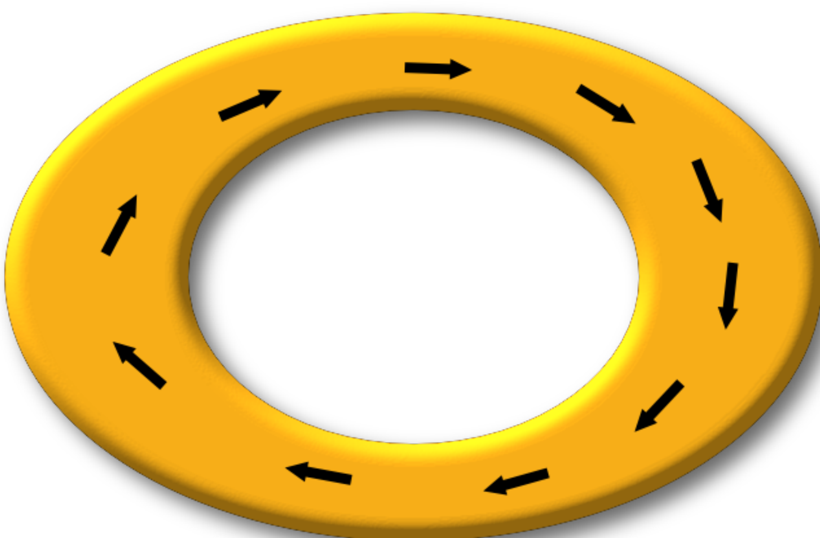
source: e-magnetsuk.com

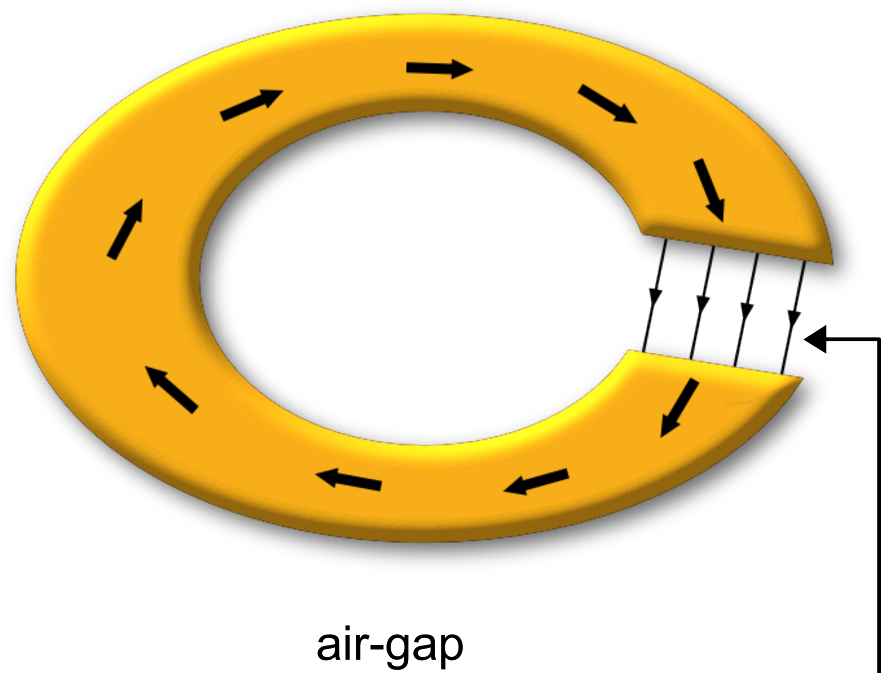
Fields reaching 5T were already obtained**.

*H.A. Shute, J.C. Mallinson, D.T. Wilton, D.J. Mapps, IEEE 2000, **36** (440)
**cerncourier.com/cws/article/cern/28598

Electromagnets

Magnetic circuits [3]:
• not all magnetic boundary problems can be solved analytically
• for some problems involving arrangements of materials of high permeability sensible first order estimates of the fields within the regions of interest can be found without knowing the analytic solution
• the problems involve multiply connected tubular regions of high magnetization so that the **outside field can be neglected at first**
• the problem can be extended to systems with small air-gaps – neglecting fringing fields





Electromagnets

The high permeability material is used to produce a magnetic field in a narrow gap [8,9]:
- the material is magnetized by a current carrying wire, wound N times around the core
- for simplicity we assume that core is a torus of major radius equal to *r*, the gap width is *d*.

From the assumptions of the previous slide (no flux leakage) it follows that (*C*-core, *G*-gap):

Ampere's law:

$$\oint_{\text{closed curve}} \vec{B}(\vec{r})\cdot d\boldsymbol{l} = \mu_0 I$$

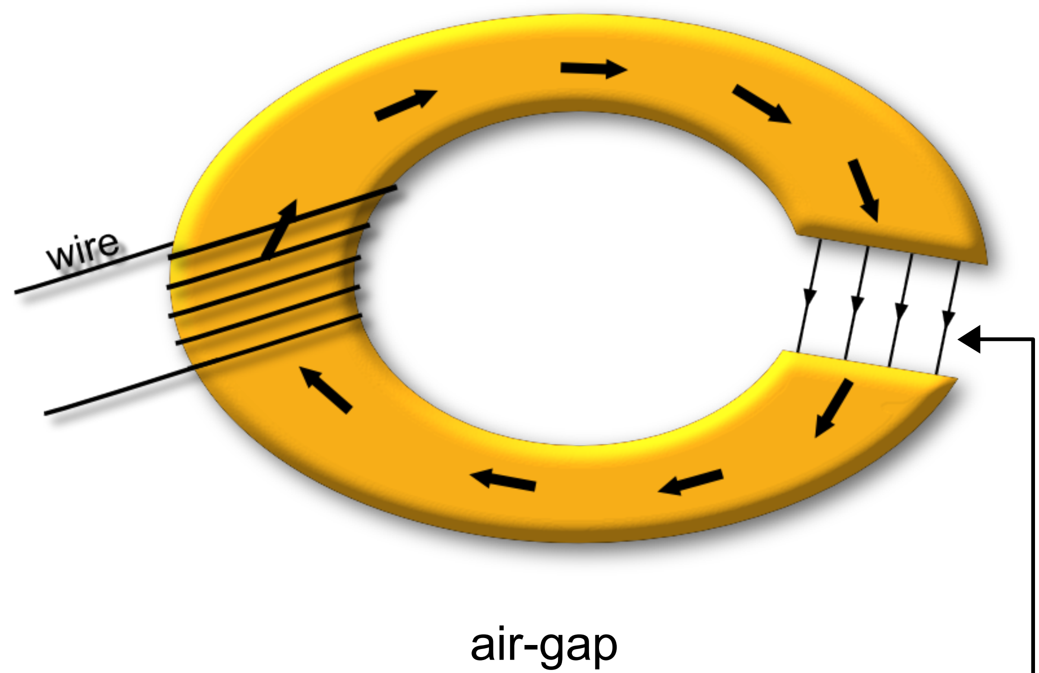
$$B_C = B_G$$

From Ampere's law we have:

$$\oint \vec{H}(\vec{r})\cdot d\boldsymbol{l} = \underbrace{(2\pi r - d)H_C}_{\text{in the core}} + \underbrace{d\,H_G}_{\text{in the gap}} = N\,I$$

Using $\vec{H}_C = \frac{1}{\mu}\vec{B}_G,\quad \vec{H}_G = \frac{1}{\mu_0}\vec{B}_G$ we get:

$$B_g = \frac{N\,I}{\frac{1}{\mu}(2\pi r - d) + d\,\frac{1}{\mu_0}}$$



Electromagnets

- The magnetic induction of typical laboratory electromagnets is about 2 T.
- The high field magnets use no magnetic core- instead high currents produce fields (Bitter electromagnets)
- World record for the magnetic field produced by a nondestructive electromagnet is **97.4 T** (set on August 23, 2011) at Los Alamos [previous record:91.4 Tesla (Dresden, June 2011)] – three seconds span.
- The strongest man-made magnetic field* **~2800 T** (Russia, 2003) – imploding magnets – very short duration (*Magnetic Flux Compression Generator*)

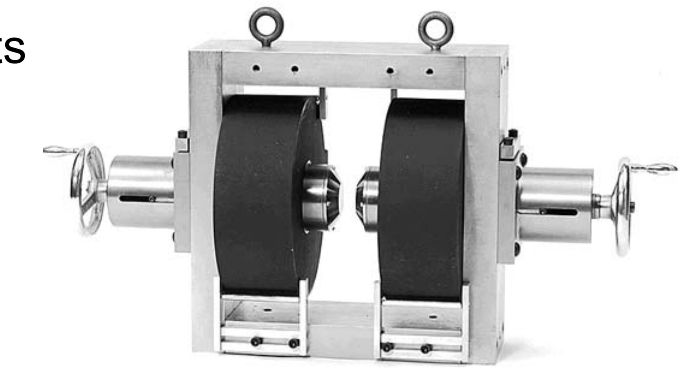


image source: appliedmagnetics.com

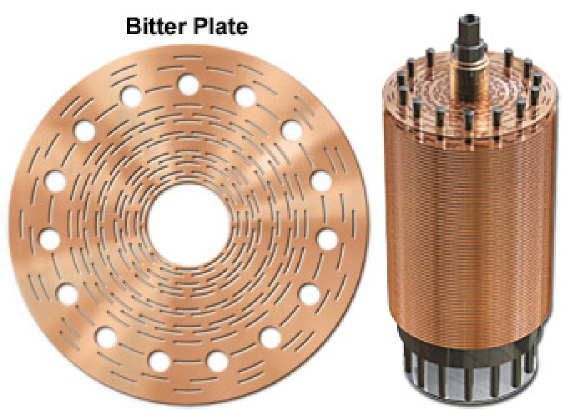


image source: National High Magnetic Field Laboratory
The Florida State University, USA, www.magnet.fsu.edu

A live frog levitates inside the Ø32mm vertical bore of a Bitter solenoid in a magnetic field of about 16 T at the Nijmegen High Field Magnet Laboratory

source Wikimedia Commons; author Lijnis Nelemans

*C.M. Fowler, L.L. Altgilbers, Электромагнитные Явления **3**, 306 (2003) http://emph.com.ua/11/pdf/fowler.pdf

Special sources of magnetic fields- examples

Field of electron beam:

- Biot-Savart field
- ~1 μm², ~10¹⁵ Am⁻²
- beam parameters (σ_x=3.5μm, σ_y=0.2μm, σ_z=9mm)
- 2-10 ps

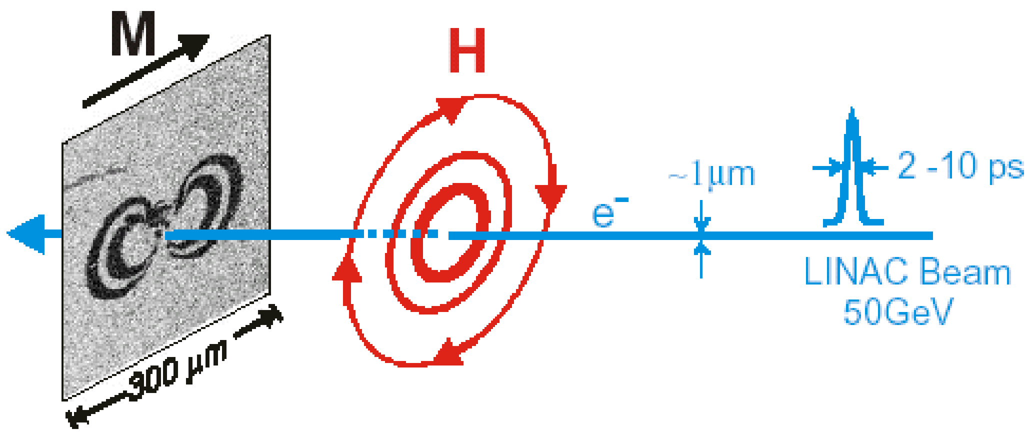


Fig. 1: Principle of the experiment with the SLAC FFTB*. The highly relativistic electron bunch generates magnetic field lines in the laboratory frame that are **equivalent to the ones from a straight current carrying wire**.

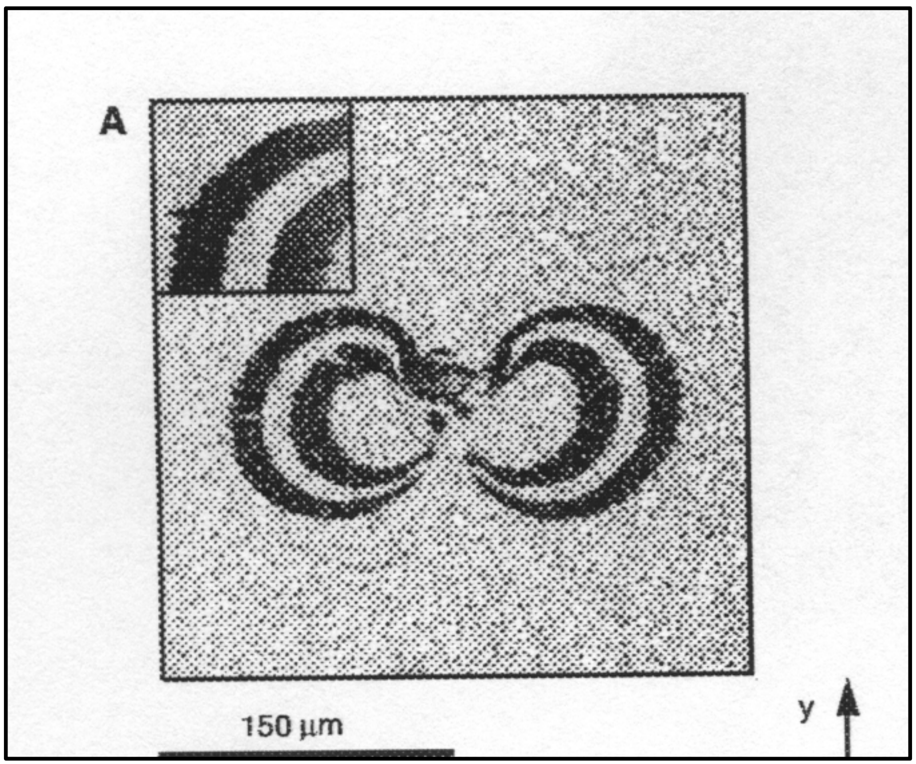
*Stanford Linear Accelerator Center (SLAC) Final Focus Test Beam (FFTB)

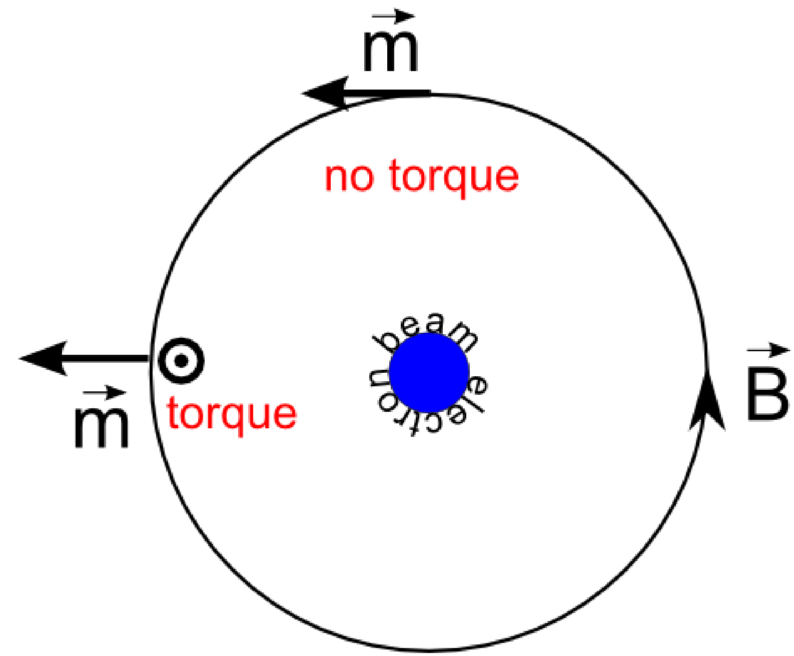
C.H. Back, R.Allenspach, W.Weber, S.S.P. Parkin, D. Weller, E.L. Garwin, H.C. Siegmann, Science 1999, **285**

Special sources of magnetic fields- examples

Field of electron beam:

- initially magnetization points in *x*-direction
- on the line with zero magnetic torque (y=0) no switching (see $\vec{N}=\vec{m}\times\vec{B}(\vec{r})$ later in this talk)





C.H. Back, R.Allenspach, W.Weber, S.S.P. Parkin, D. Weller, E.L. Garwin, H.C. Siegmann, Science 1999, **285**

Special sources of magnetic fields- examples

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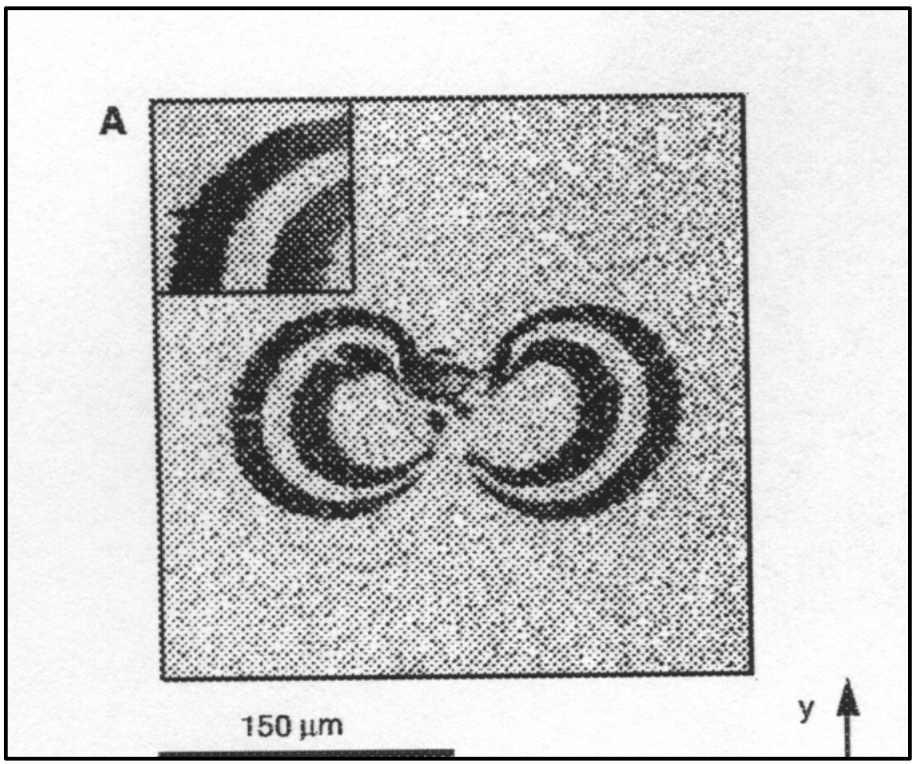
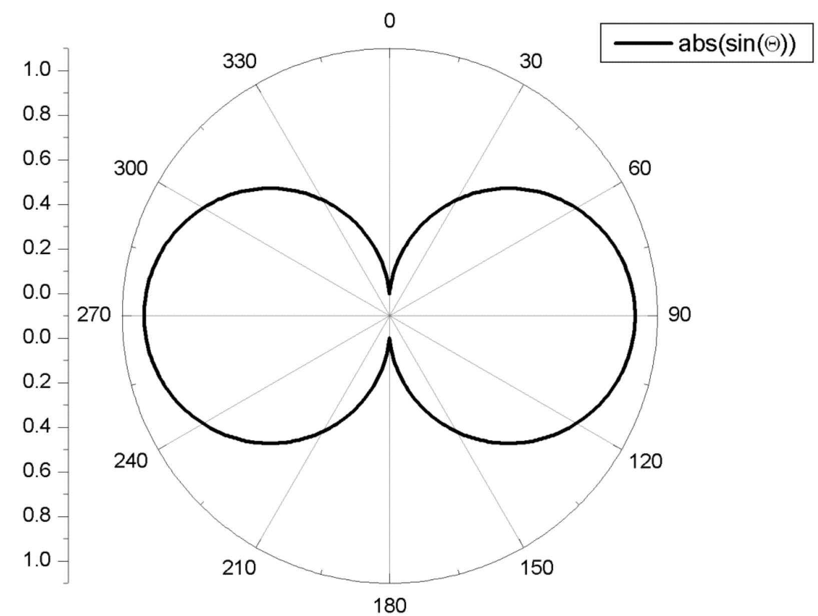




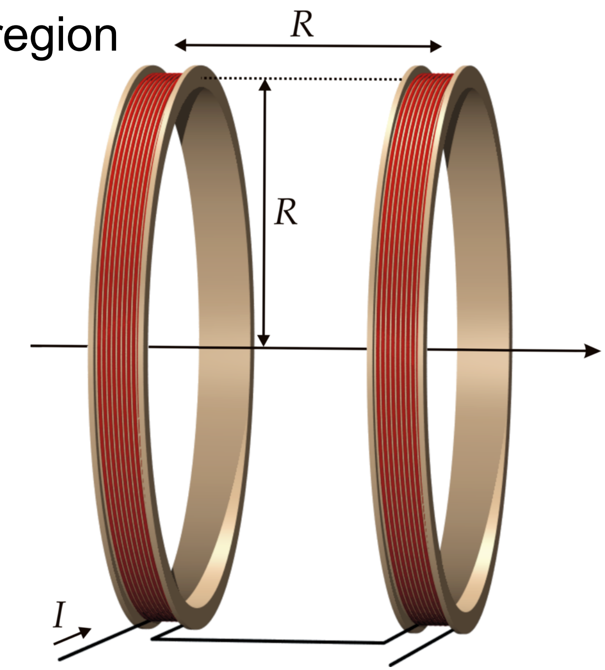
$$\vec{N}=\vec{m}\times\vec{B}(\vec{r})$$

C.H. Back, R.Allenspach, W.Weber, S.S.P. Parkin, D. Weller, E.L. Garwin, H.C. Siegmann, Science 1999, **285**

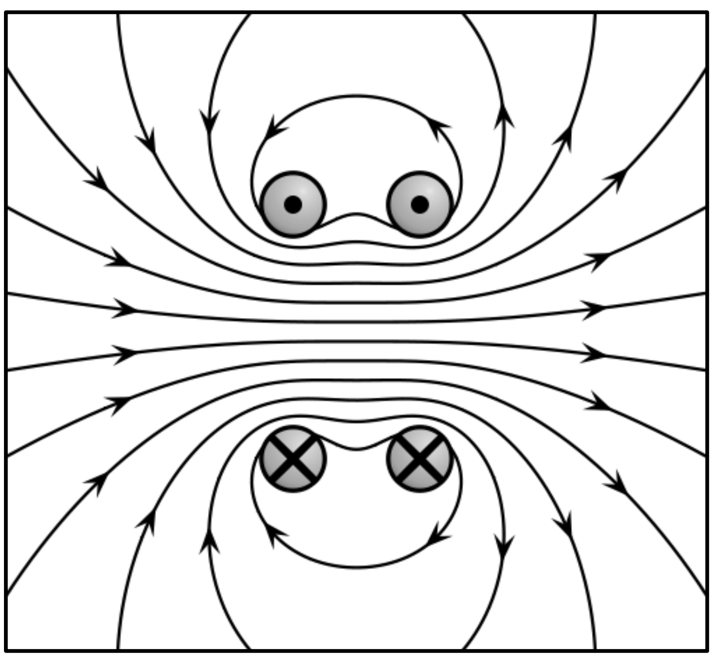
Special sources of magnetic fields- examples

Helmholz coils:

- to obtain nearly uniform field (usually weak) within a large region
- coils placed apart a distance equal to their radii
- each coil carries equal current



source: Wikimedia Commons; author Ahellwig

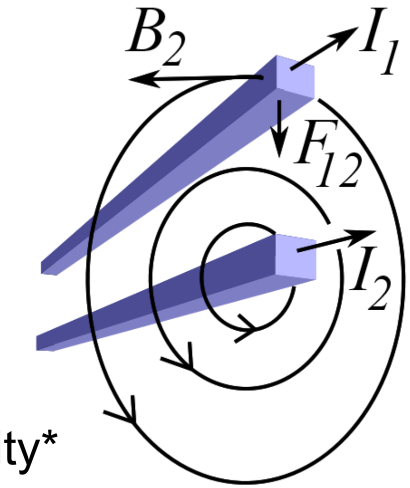


source: Wikimedia Commons; author Geek3

Force between two current carrying wires

There is a force acting on a moving electric charge placed in magnetic field:

$$\boxed{\vec{F}_{Lorentz} = q\vec{E} + q\vec{v}\times\vec{B}}$$ Lorentz force



source: Wikimedia Commons

The magnetic force acting on the volume element carrying current is [4]:

$$d\vec{F} = \varrho_V d^3r\,\vec{v}\times\vec{B} = \vec{j}_V\times\vec{B}\,d^3r$$

ϱ_V - volume charge density*

The overall force is obtained by the integration:

$$\vec{F} = \int_V \vec{j}_V\times\vec{B}\,d^3r$$

Integrating that expression for two infinite, parallel, straight wires gives the expression for the attraction force (if currents in both of them flow in the same direction) per unit length:

$$\boxed{\vec{F} = \mu_0\frac{I_1 I_2}{2\pi d}}$$

This equation is the basis of *Ampere* definition.

Or from field of straight wire (L.1):

$$B = \frac{\mu_0 I}{2\pi r},\quad q\cdot v = (S\rho)\frac{I_2}{S\rho} = I_2 \;\Rightarrow\; F = \mu_0\frac{I_1 I_2}{2\pi r}$$

cross section of wire

electron charge density

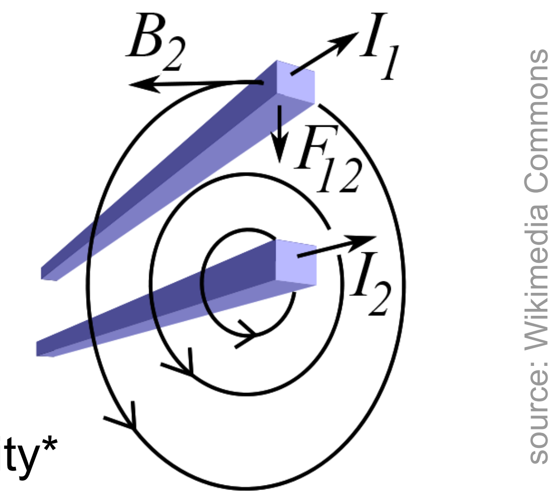
*local density of electric charge is usually zero so that electrostatic interaction is negligible

Force between two current carrying wires

There is a force acting on a moving electric charge placed in magnetic field:

$$\vec{F}_{Lorentz}=q\vec{E}+q\vec{v}\times\vec{B}$$

Lorentz force



The magnetic force acting on the volume element carrying current is [4]:

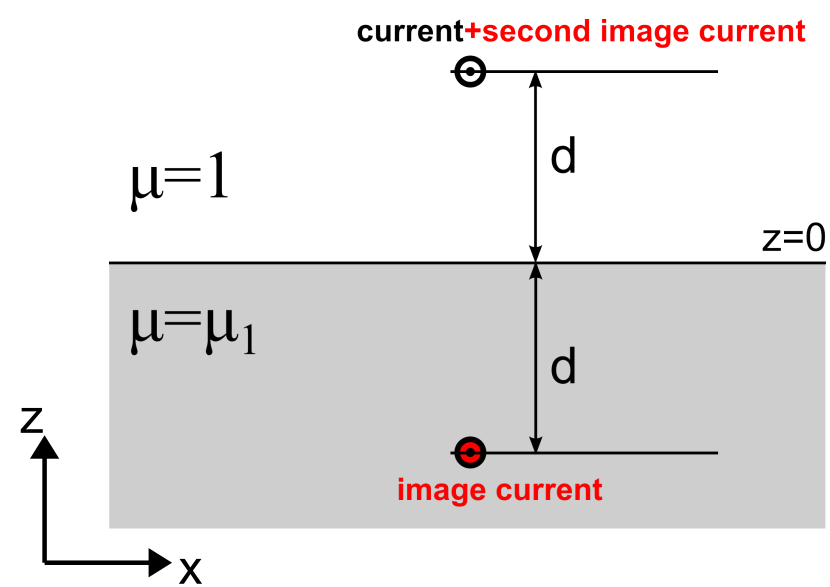
$$d\vec{F}=\varrho_V d^3r\,\vec{v}\times\vec{B}=\vec{j}_V\times\vec{B}\,d^3r$$

ϱ_V - volume charge density*

The force between current and magnetic body:
-the force between current and its image current

$$\frac{\mu-1}{\mu+1}J_x(x,y,-z),\qquad \frac{\mu-1}{\mu+1}J_y(x,y,-z),\qquad -\frac{\mu-1}{\mu+1}J_z(x,y,-z)$$

$$\vec{F}=\mu_0\frac{I_1 I_2}{2\pi d}$$



Torque on a magnetic dipole

The torque is calculated from the general expression:

$$\vec{N}=\int_V \vec{r}\times\vec{j}_V(\vec{r})\times\vec{B}(\vec{r})\,d^3r$$

This time however already the first term of ***B*** expansion gives nonvanishing term:

$$\vec{N}=\vec{m}\times\vec{B}(\vec{r})$$

$$\vec{F}=\nabla(\vec{m}\cdot\vec{B})$$

From each of these two equations it follows that the potential energy of a dipole in magnetic field can be expressed as:

$$E=-\vec{m}\cdot\vec{B}$$

- The above expression does not in general describe the total energy of a dipole; placing the moment in magnetic field requires the additional energy with which the current source maintains the magnitude of the moment under the influence of magnetic (Faraday) induction.
- In case of elementary particles with the spin (electron, neutron etc) their intrinsic magnetic moment is constant and the above expression gives the total energy.

Things to remember from today's talk:

- Magnetic charges although not physical are useful in solving magnetostatic problems
- Biot-Savart law and magnetic charges methods are equivalent
- Demagnetizing fields originate from magnetic charges of the magnetized body itself; they diminish magnetic field within ferromagnets
- You do not need to use the idea of magnetic charges to calculate the field inside the magnet – amperian approach

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