1

Sources of Magnetic Field

MAGNETIC MATERIALS AND HYSTERESIS

Poznań 2019

Maciej Urbaniak

1
Sources of Magnetic Field

- Introduction
- The beginnings of the science of magnetism
- The field of the currents - Biot-Savart law
- The field of magnetic dipoles
- Magnetization
- Sources specific for small scale devices

Importance of “magnetic” elements

2016 World production of some elements [23] (production figures do not refer to crude ore or concentrate produced from it, “*but indicate the content of recoverable valuable elements and compounds*”)

Element	Production [metr. t]
Iron	1 575 123 716
Nickel	1 953 503
Cobalt	126 234
Gold	3 214
Platinum	185

[23] World Mining Data 2018, Bundesministerium Für Nachhaltigkeit Und Tourismus, Wien, Austria, www.bmnt.gv.at

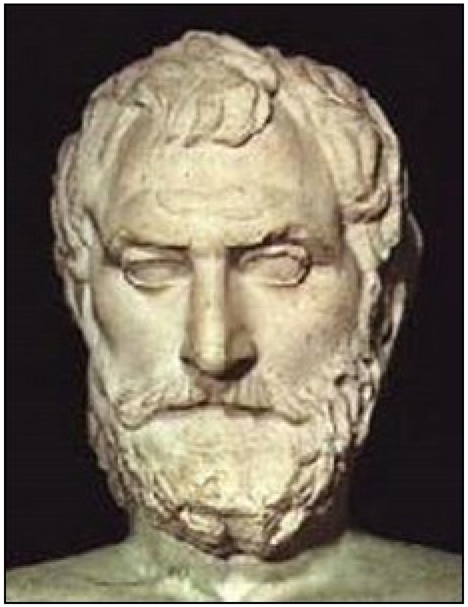
Importance of “magnetic” elements

- Note that although Fe is a major component in most magnetic alloys* it is used predominantly (in steels) in construction and automobile industries (35% + 19% for EU countries [24, 25 for global data])
- In EU only about 2.3% [24, p.26] of manufactured steel is an electrical steel; assuming the similar relation in global perspective** and pretending that steel is 100% iron we get roughly 36 mln t of steel used globally for its magnetic properties (electric drives, transformers etc.) every year.
- Of all hard ferrite magnets about **45 wt%** are used for production of motors (automotive, appliance, industrial etc.), some 11 wt% for manufacturing of loudspeakers [26]

*see Table 1 in an editorial of spring 2016 issue of Magnetics magazine (https://magneticsmag.com/permanent-magnets-in-a-changing-world-market/) **I have no access to exact data

A bit of history

Thales of Miletus (about 585BCE)- the first mention of the influence of loadstone* on iron [1]



Aristotle: ‘Thales, too, to judge from what is recorded of his views, seems to suppose that the soul is in a sense the cause of movement, since he says that a **stone** [magnet, or lodestone] **has a soul** because it causes **movement to iron**’ (On the soul (Περὶ Ψυχῆς, Perì Psūchês), 405 a20-22)

source: Alokprasad at en.wikipedia; cropped

Probably the first practical application of magnetism:

Sushruta Samhita (Indian book from IV century CE giving supposedly teachings of surgeon Sushruta acting about 600 BCE):
A loose, unbarbed arrow, lodged in a wound with a broad mouth and lying in an Anuloma direction, **should be withdrawn by applying a magnet** to its end.

*alternative spelling: lodestone

Lucretius (98-55 BCE)- the first recorded theory of magnetic interactions (following the view of Epicurus and Democritus [1]. De rerum natura (O naturze wszechrzeczy, translation in polish E. Szymański):

Teraz powiem, na mocy jakiego natury prawa
Może żelazo przyciągać **ten kamień, który Greki**
Magnesem zwą od ziemi Magnetów — w tym bo dalekim
Kraju kamień ten cenny rodzi się i przebywa.
Ludzi uczonych od dawna nie darmo on zadziwia:
…
Teraz cel osiągniemy dokładniej już i prędzej.
Bo skoro wszystkie dane sprawdzone i gotowe,
Z ich pomocą prawdziwie poznamy siły owe,
Dzięki którym kamień żelazo do siebie przyzywa.
Naprzód musi z kamienia dużo ziaren wypływać.
Istny prąd, co roztrąca swem mocnem uderzeniem
Warstwę powietrza między żelazem i kamieniem.
Gdy się opróżni przestrzeń i w środku miejsca sporo,
Zaraz ziarna żelaza wyskoczą, wnet się zbiorą
Próżnię wypełnić, zaczem zbliża się i ogniwo,
Całem swem ciałem dążąc ku kamieniowi co żywo.

”In other word, tiny particles emanating from the loadstone sweep away the air and the consequent suction draws in the iron” -Fowler [1]

Lucretius:

- the gold is to heavy to be attracted by magnets
- the wood is so light...

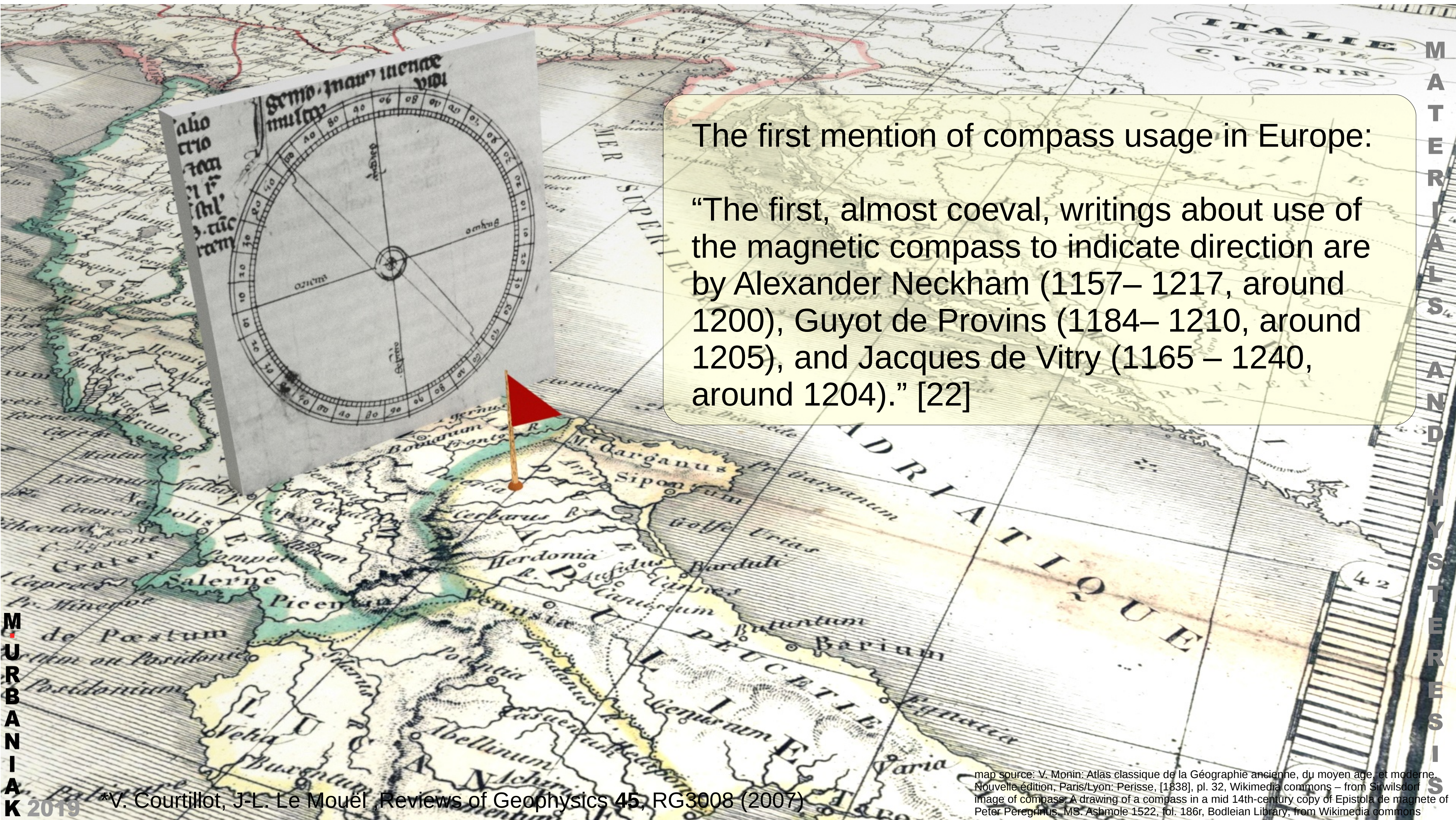
A bit of history

And of course there were Chinese. They new magnetic needle from ca. 400 BCE. But the first Chinese mention of the use of magnetic needle for navigation refers to the period 1086-99 and concerns the use by ”*Muslim sailors between Canton and Sumatra*” [5].

The South-Pointing Fish

A bit of history

Petrus Peregrinus (the pilgrim) de Maricourt (**13th century**, active 1261-1269) –,,*was a* [french] *knight from Picardie who wrote his small masterpiece letter on the magnet on 8 August 1269, while participating in the siege of the Italian town of Luceria (or Lucera) by Charles of Anjou*" [22*]



The first mention of compass usage in Europe:

"The first, almost coeval, writings about use of the magnetic compass to indicate direction are by Alexander Neckham (1157– 1217, around 1200), Guyot de Provins (1184– 1210, around 1205), and Jacques de Vitry (1165 – 1240, around 1204)." [22]

*V. Courtillot, J-L. Le Mouël ,Reviews of Geophysics **45**, RG3008 (2007)

map source: V. Monin: Atlas classique de la Géographie ancienne, du moyen age, et moderne. Nouvelle édition, Paris/Lyon: Perisse, [1838], pl. 32, Wikimedia commons – from Sirwilsdorf
image of compass: A drawing of a compass in a mid 14th-century copy of Epistola de magnete of Peter Peregrinus. MS. Ashmole 1522, fol. 186r, Bodleian Library; from Wikimedia commons

A bit of history

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- "*the most important advance in knowledge of magnetic properties to emerge from Europe since the discovery of lodestone by the Greeks*" [Ref. 22 citing P. Smith, Petrus Peregrinus epistola—The beginning of experimental studies of magnetism in Europe, Earth Sci. Rev. 6, A11 (1970)**]

Peregrinus writes about [22]:

- magnetizing action of lodestone on the iron
- magnetic poles, the interaction between them and the fact that broken pieces of magnet still have two opposite poles
- describes the working of pivoted (dry) and floating compass (he was unaware of the declination and rejected the idea that the earth was a source of the magnetic field)

*V. Courtillot, J-L. Le Mouël, Reviews of Geophysics **45**, RG3008 (2007)

** I did not have access to the Smith's paper

William Gilbert (1544-1603) – royal physician to Queen Elizabeth I

-”De Magnete” (1600) “the first” scientific investigation of magnetism [1]:

- **the earth is a giant magnet** (previously there was a belief that there was a magnetic island or star ***Polaris*** that attracted compass needles)
- magnetic (and electric) attraction depends on **the distance between bodies**

Working iron in a smithy

A bit of history

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- **the earth is a giant magnet** (previously there was a belief that there was a magnetic island or star ***Polaris*** that attracted compass needles)
- magnetic (and electric) attraction depends on **the distance between bodies**

south

inducing magnetic anisotropy by metalworking

Earth magnetic field orients the elementary magnets within the piece of metal

north

Working iron in a smithy

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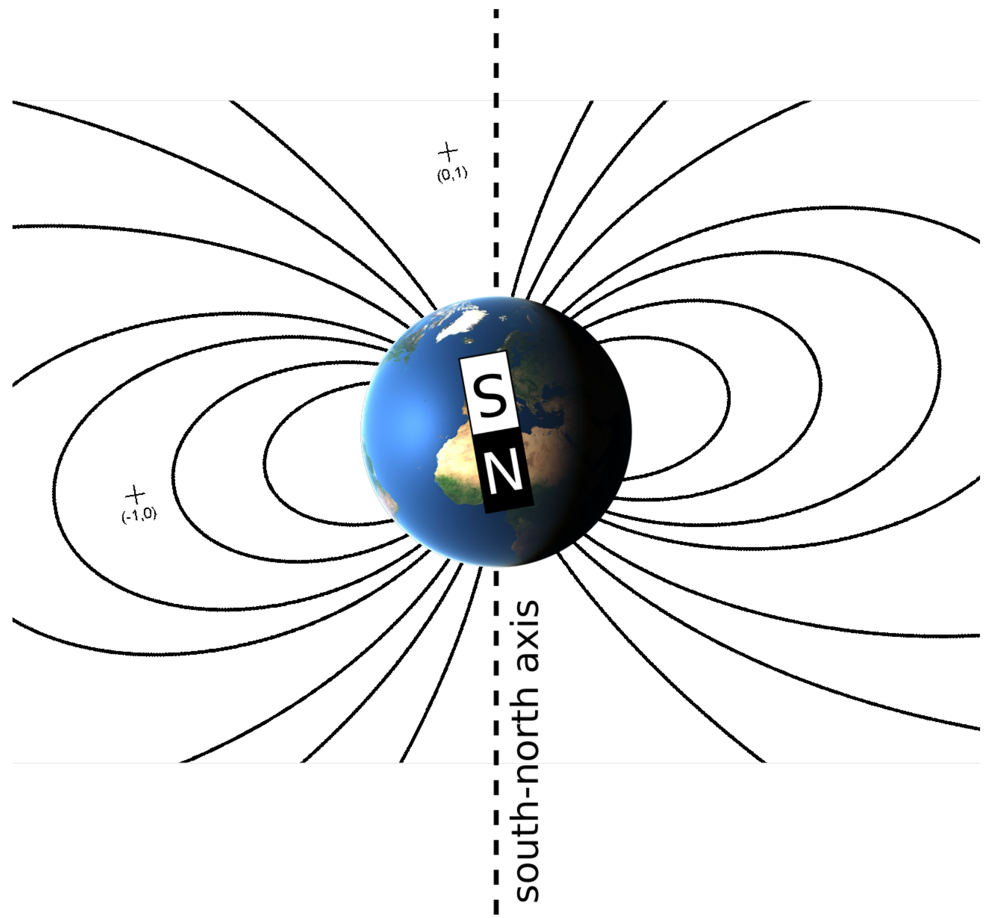


Note that earth magnetic north pole is physically a south pole

Jacques Rohault (1620 – 1675), *Traité de Physique*, 1671 [22]

- using iron filings he shows the direction of field lines of a magnet
- shows that a piece of steel can be magnetized by cooling it in Earth's field – thermal remanent magnetization (TRM)
- shows that heating the magnet destroys its magnetization

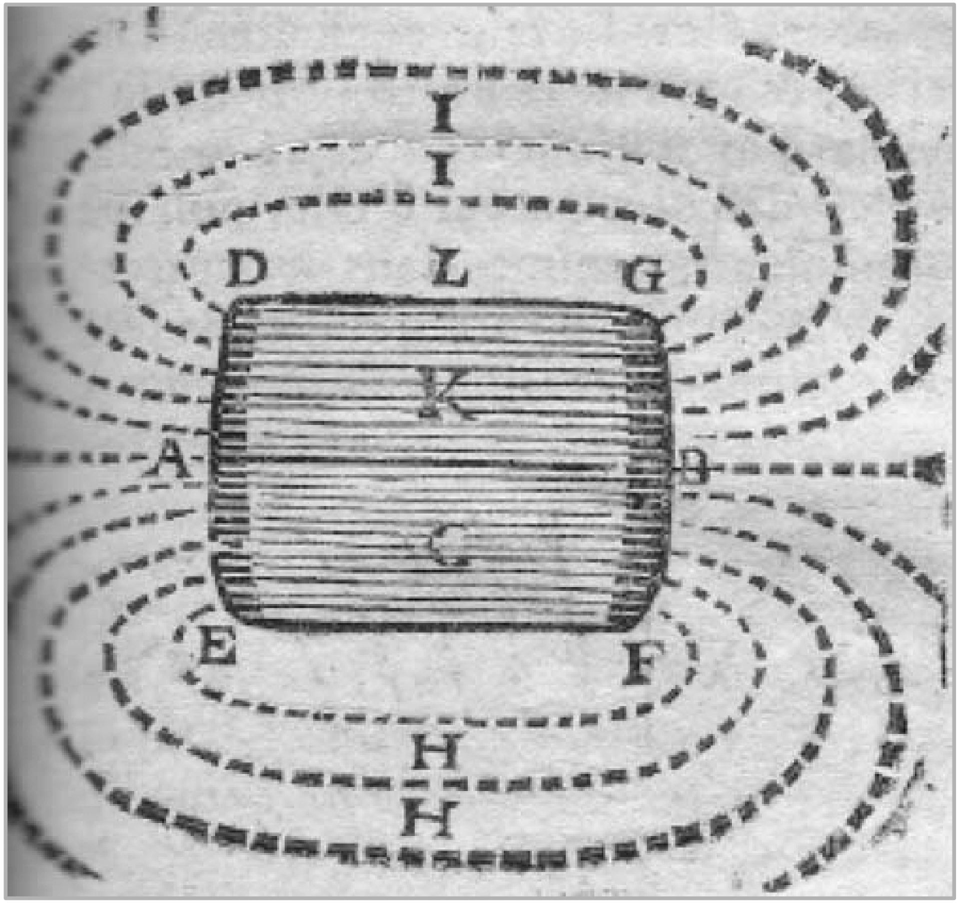
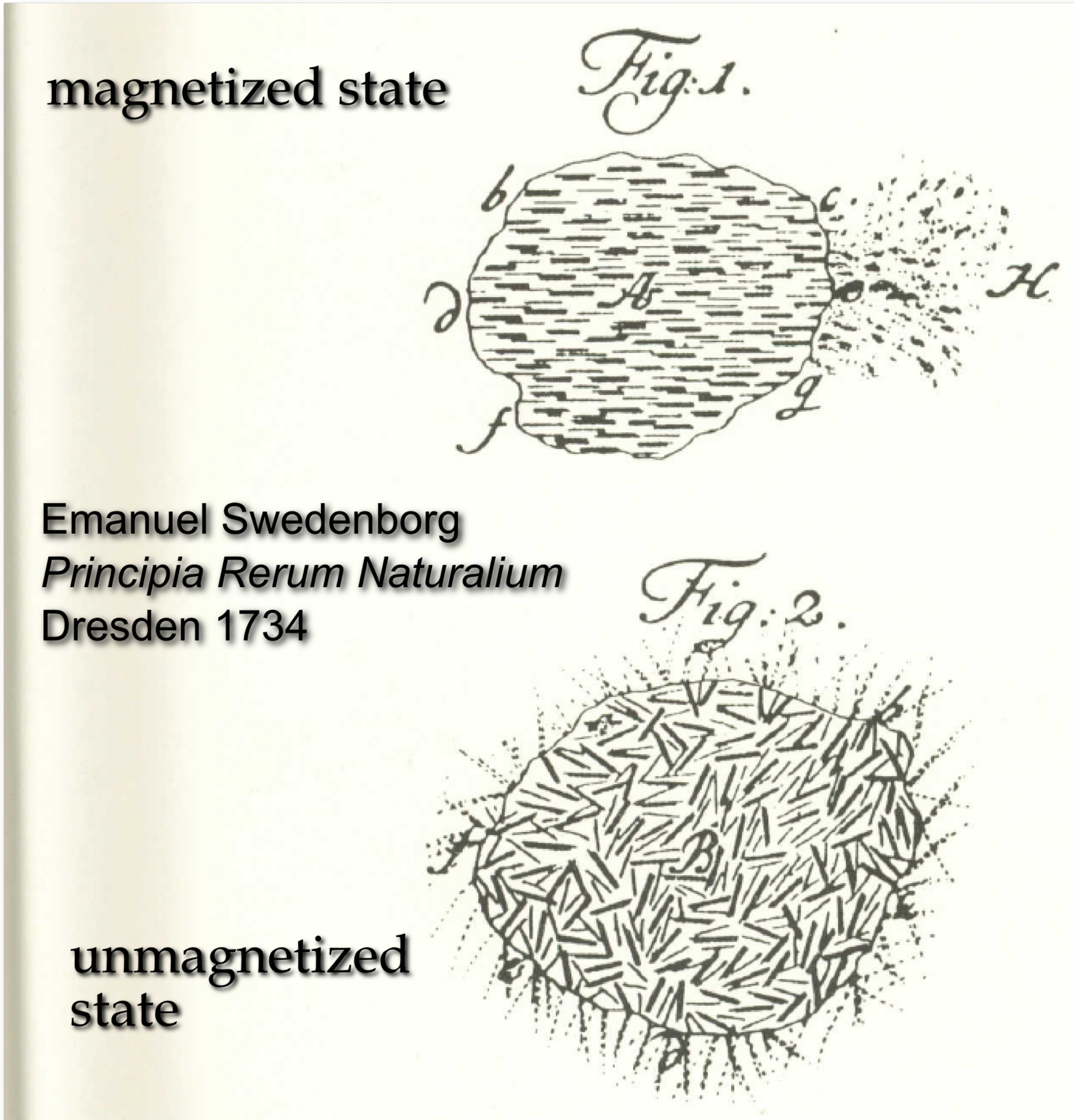


image source:
Traité de Physique, J. Rohault, 1682 as reproduced in [22]: V. Courtillot, J-L. Le Mouël, Reviews of Geophysics **45**, RG3008 (2007)

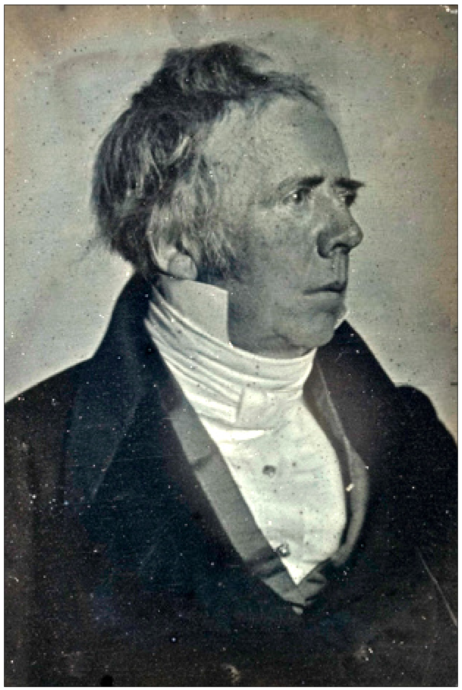
A bit of history   Magnetic domains – early views



original image taken from:
B. D. Cullity
Introduction to magnetic materials
Addison-Wesley, Reading,
Massachusetts 1972

A bit of history

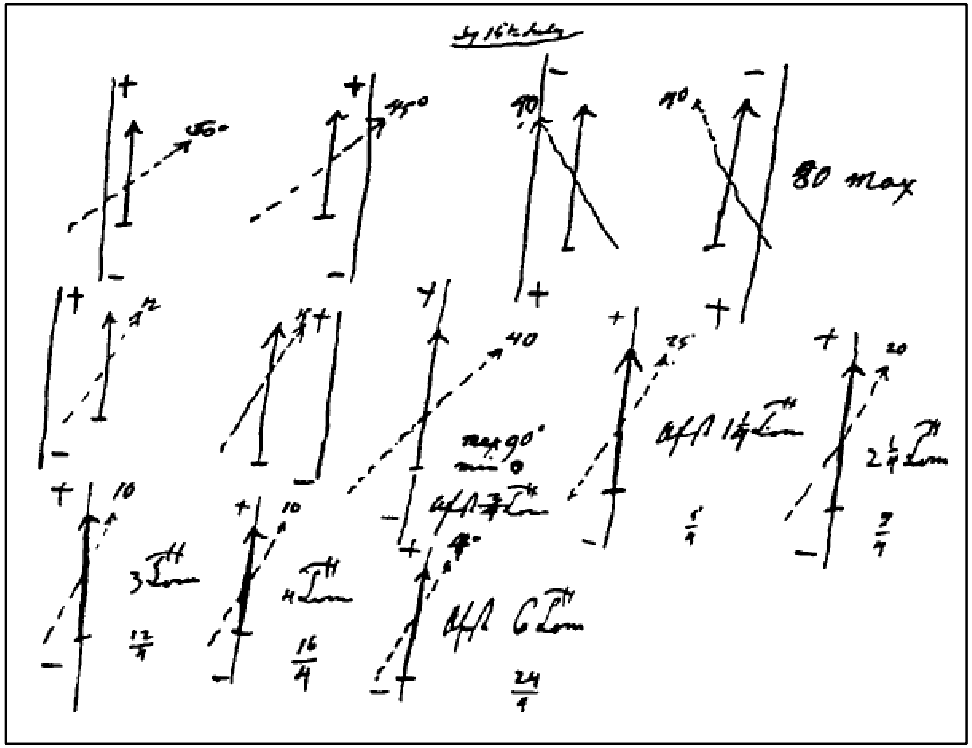
Hans Christian Ørsted (1777–1851)

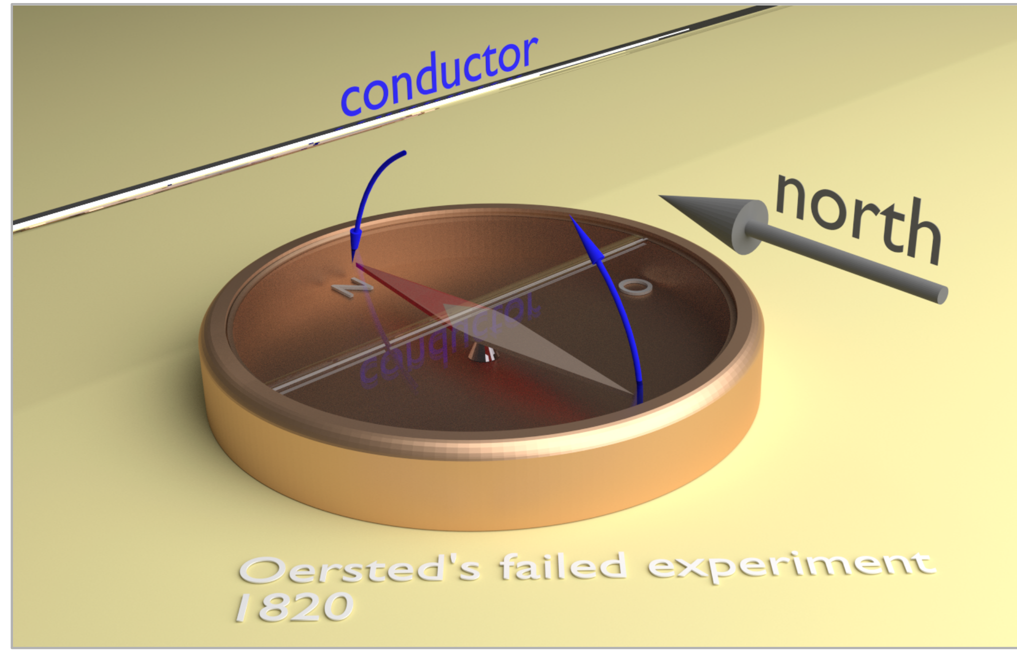


- Around 1750 Benjamin Franklin magnetized sewing needles by an electrical discharge of a Leyden jar [6] but the effect was due to Joule heating in the Earth's magnetic field.
- In 1795 Coulomb established that magnetic forces obey the inverse square law [6].
- In 1805 Hachette and Désormes unsuccessfully attempted to build a electric compass [6].
- In 1820 Ørsted discovers that electric **current deflects magnetic needle** – the begin of electromagnetism.

A bit of history

Hans Christian Ørsted (1777–1851)



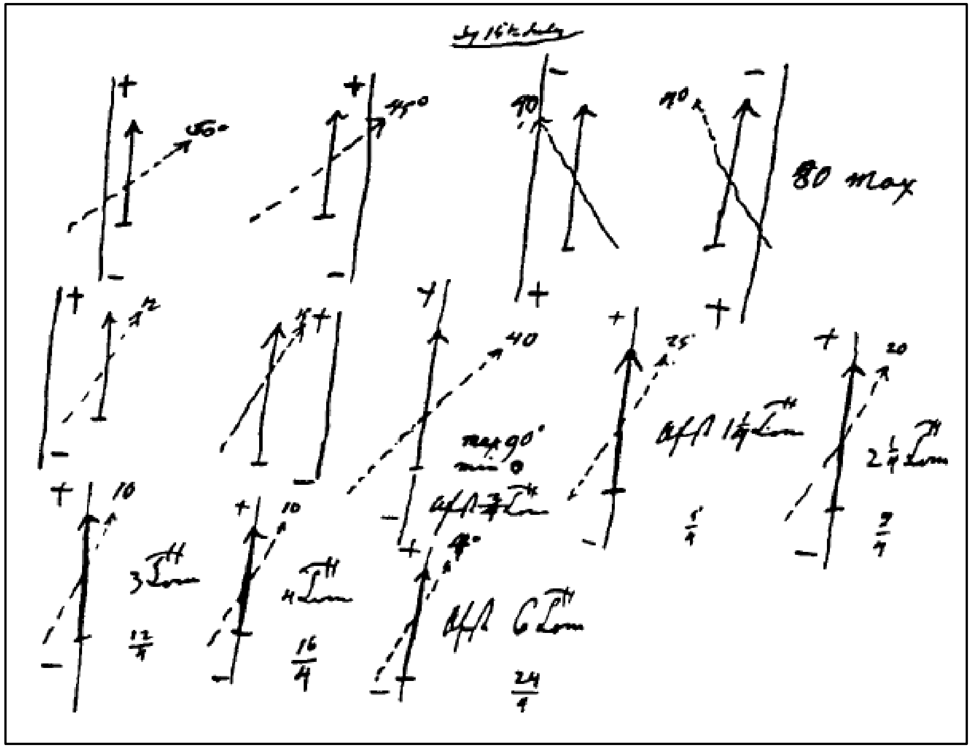


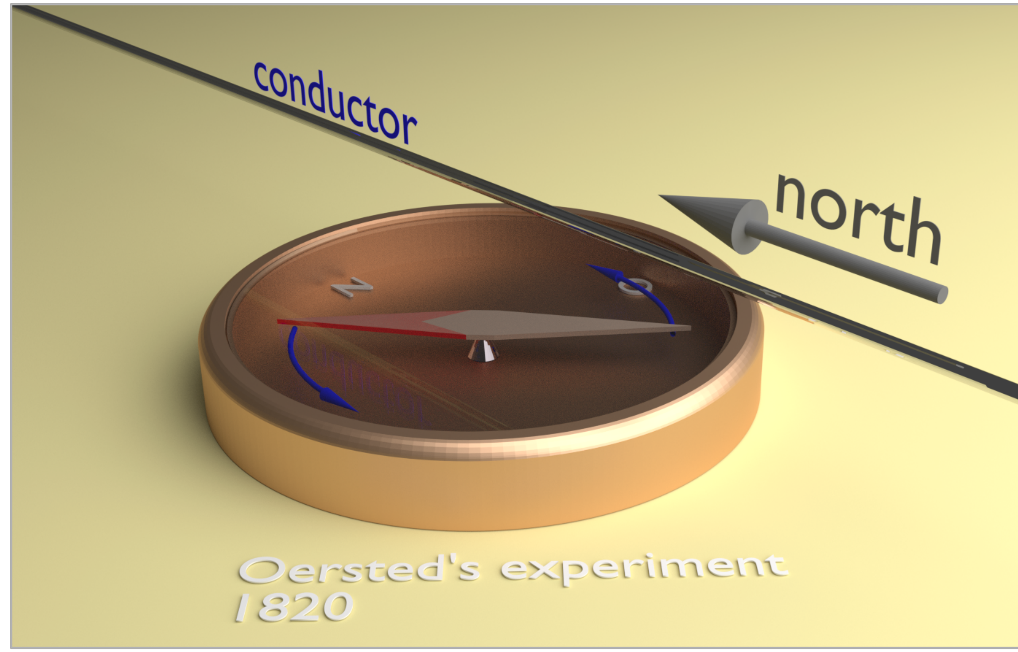
Ørsted's laboratory notes from 1820.07.15

- Before 1820 Ørsted’s first hypothesis was that the magnetic effect should be parallel to the wire [6] – it lead to the misplacement of the wire relative to the south-north direction: a force couple would act to turn the needle in a vertical plane, and the suspension of the needle would prevent this kind of motion. So, if Ørsted attempted such experiments, he could observe no effect [6].

A bit of history

Hans Christian Ørsted (1777–1851)





Ørsted's laboratory notes from 1820.07.15

- Before 1820 Ørsted’s first hypothesis was that the magnetic effect should be parallel to the wire [6] – it lead to the misplacement of the wire relative to the south-north direction.
- According to Ørsted’s final view, the **magnetic effect of an electric current rotates around the conducting wire**

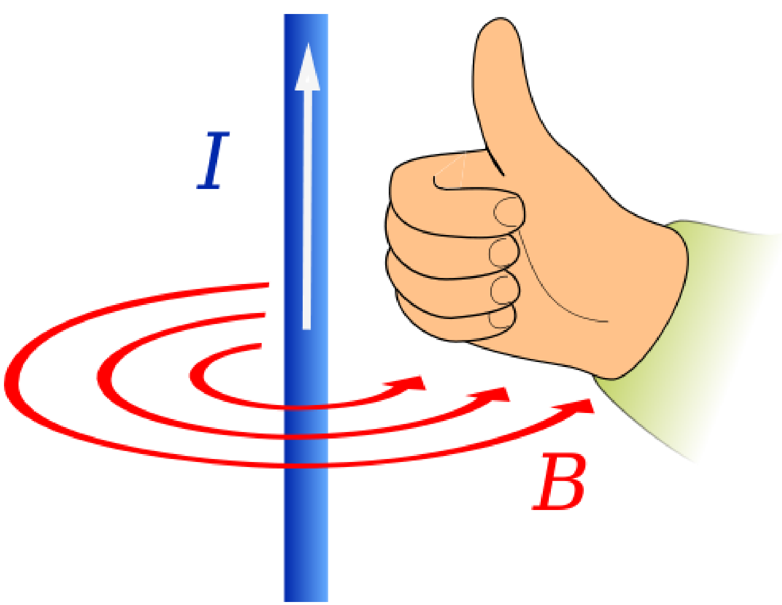
Biot-Savart law - 1820

- Jean-Baptiste Biot (1774-1862), Félix Savart (1791-1841)

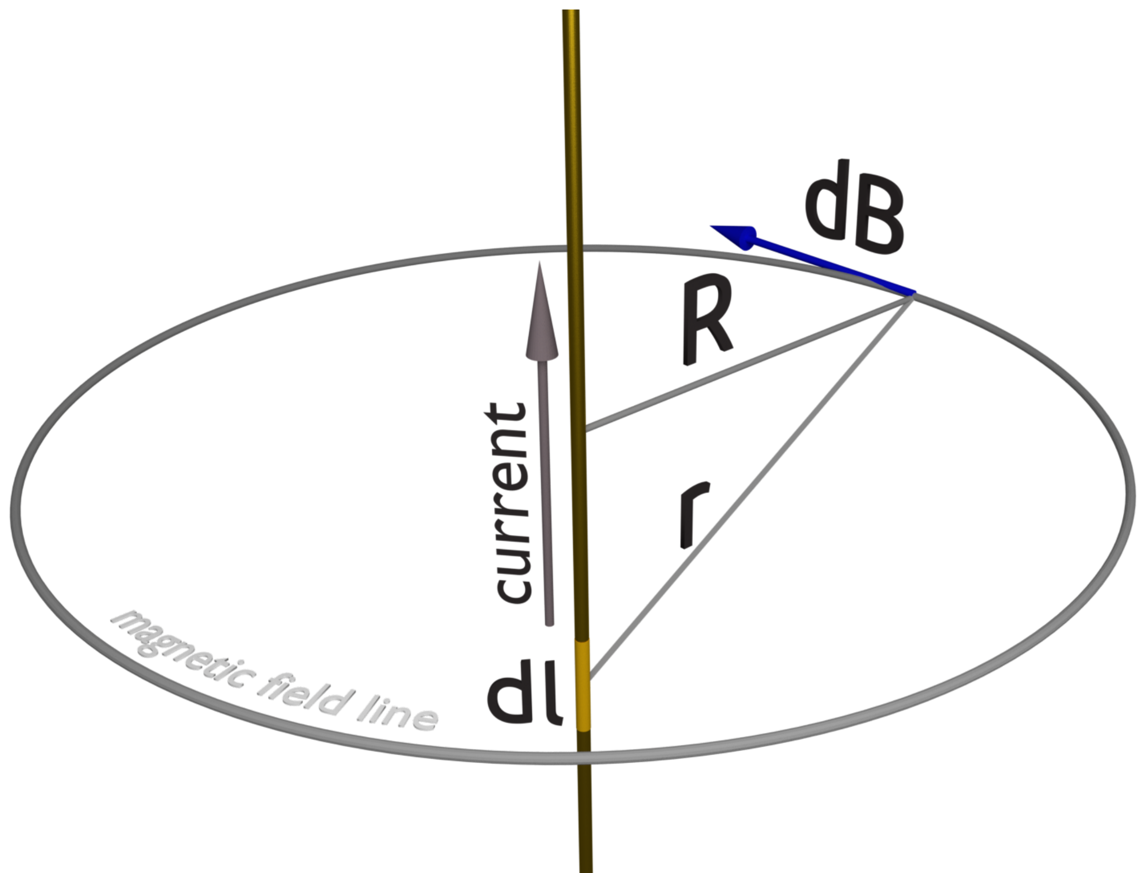
$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\hat{d}l \times \vec{r}}{|\vec{r}|^3} \qquad \mu_0 = 4\pi 10^{-7}\,\mathrm{Hm}^{-1}$$

-vacuum permeability

- magnetic field is created by the electric current
- meaningful only for closed circuits



author: Jfmelero; from Wikimedia Commons



Biot-Savart law - 1820

$$d\vec{B}=\frac{\mu_0 I}{4\pi}\frac{\hat{dl}\times\vec{r}}{|\vec{r}|^3}=\frac{\mu_0 I}{4\pi}\frac{dy\,|\vec{r}|\sin(\theta)}{|\vec{r}|^3}=\frac{\mu_0 I}{4\pi}\frac{dy\,|\vec{r}|\frac{R}{|\vec{r}|}}{|\vec{r}|^3}=\frac{\mu_0 I}{4\pi}\frac{R\,dy}{|\vec{r}|^3}=\frac{\mu_0 I}{4\pi}\frac{R\,dy}{\left(\sqrt{R^2+y^2}\right)^3}$$

The problem has a circular symmetry so the magnitude of ***B*** depends only on R.

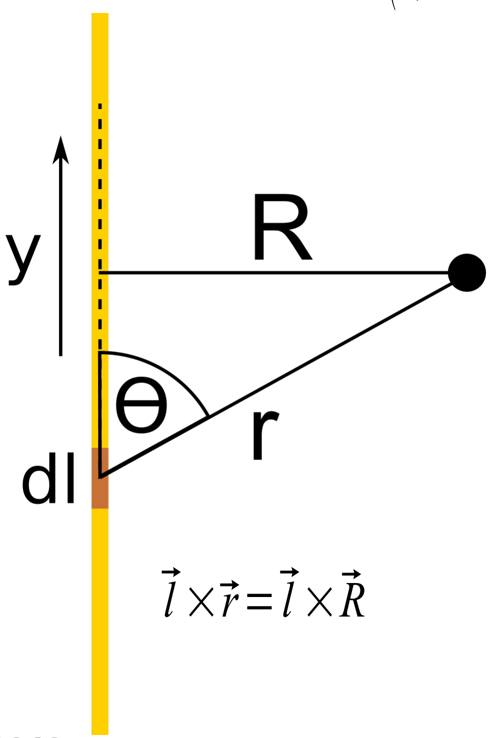
$$\vec{B}(R)=\frac{\mu_0 I R}{4\pi}\int_{-\infty}^{\infty}\frac{dy}{\left(\sqrt{R^2+y^2}\right)^3}=\frac{\mu_0 I R}{4\pi}\left[\frac{y}{R^2\left(\sqrt{R^2+y^2}\right)}\right]_{-\infty}^{+\infty}=\frac{\mu_0 I}{2\pi R}$$

$$\lim_{y\to\infty}\frac{y}{\left(\sqrt{R^2+y^2}\right)}=\lim_{y\to\infty}\sqrt{\frac{y^2}{R^2+y^2}}=\lim_{y\to\infty}\sqrt{\frac{1}{R^2/y^2+1}}=$$

$$\sqrt{\lim_{y\to\infty}\frac{1}{R^2/y^2+1}}=\sqrt{\frac{1}{1}}=1$$

$$\vec{B}(R)=\frac{\mu_0 I}{4\pi R}\left[1-(-1)\right]$$

$$\boxed{\vec{B}(R)=\frac{\mu_0 I}{2\pi R}}$$



$$\vec{l}\times\vec{r}=\vec{l}\times\vec{R}$$

- An infinite straight conductor carrying a current of 1 A creates a magnetic field which is weaker than earth's magnetic field (~10^{-5} T) at a distance greater than **4 millimeters** from the wire.
- Passing a current through a straight wire is not an effective way of generating magnetic field [11].

Basic properties of static magnetic field

It follows from Biot-Savart law that [7,8]:

$$d\vec{B}=\frac{\mu_0 I}{4\pi}\frac{\hat{dl}\times\vec{r}}{|\vec{r}|^3}$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\times\frac{r-r'}{|r-r'|^3}d^3r'$$

$\vec{J}(\vec{r}\,')$ - current density

Using the identity:

$$\nabla_{\vec{r}}\left(\frac{1}{|r-r'|}\right)=\nabla_{\vec{r}}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)=-\frac{\vec{r}-\vec{r}\,'}{|r-r'|^3}$$

We obtain:

$$\vec{B}(\vec{r})=\frac{-\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\times\nabla\left(\frac{1}{|r-r'|}\right)d^3r'$$

Using the identity $\nabla\times(\beta\vec{a})=\beta\nabla\times\vec{a}-\vec{a}\times\nabla\beta$ with $\vec{a}\rightarrow\vec{J}$ and $\beta\rightarrow 1/|r-r'|$ we get:

$$\boxed{\vec{J}(\vec{r}\,')\times\nabla\left(\frac{1}{|r-r'|}\right)=\frac{1}{|r-r'|}\nabla\times\vec{J}(\vec{r}\,')-\nabla\times\frac{\vec{J}(\vec{r}\,')}{|r-r'|}}$$

, but ***J*** does not depend on ***r***, so*...

*__r__ is the observation point and __r'__ describes the current distribution

Basic properties of static magnetic field

so...

$$\vec{J}(\vec{r}\,')\times\nabla\left(\frac{1}{|r-r'|}\right)=-\nabla\times\frac{\vec{J}(\vec{r}\,')}{|r-r'|}$$

$$\vec{J}(\vec{r}\,')\times\nabla\left(\frac{1}{|r-r'|}\right)=\frac{1}{|r-r'|}\underbrace{\nabla\times\vec{J}(\vec{r}\,')}_{0}-\nabla\times\frac{\vec{J}(\vec{r}\,')}{|r-r'|}$$

, and thus

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\nabla\times\frac{\vec{J}(\vec{r}\,')}{|r-r'|}d^3r'$$

, and since rotation operator does not act on primed coordinates we can rewrite (nabla moves outside the integral):

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\nabla\times\int\frac{\vec{J}(\vec{r}\,')}{|r-r'|}d^3r'=\frac{\mu_0}{4\pi}\nabla\times\text{some vector field} \quad (1)$$

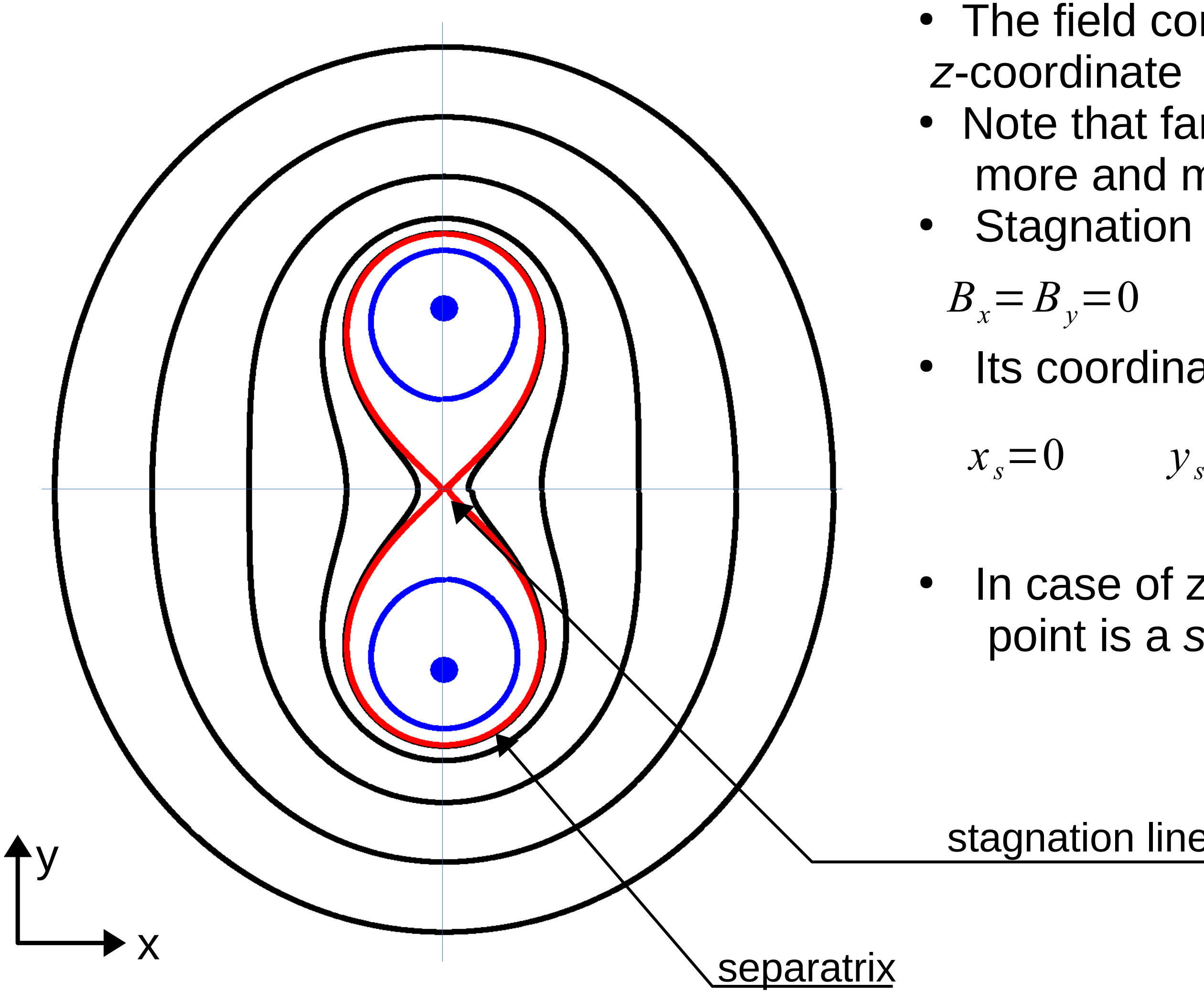
Using vector identity $\nabla\cdot(\nabla\times\vec{a})=0$ we get the first differential equation of magnetostatics:

$$\boxed{\nabla\cdot\vec{B}=0}$$

- there are no sources or sinks of magnetic induction vector (there are no magnetic charges emanating magnetic induction)
- ***B*** is a solenoidal field

Separatrix and stagnation points

- Directions of magnetic field of two parallel, infinite currents lines:



two currents of the same direction and magnitude

- The field configuration does not depend on *z*-coordinate
- Note that far from currents the field lines are more and more circle-like
- Stagnation point is defined by [19]:

$$B_x = B_y = 0$$

- Its coordinates are:

$$x_s = 0 \qquad y_s = \frac{d}{2}\frac{I_1 - I_2}{I_1 + I_2}, \quad \text{d-spacing of wires}$$

- In case of z-independent field the stagnation point is a *stagnation line*

Basic properties of static magnetic field

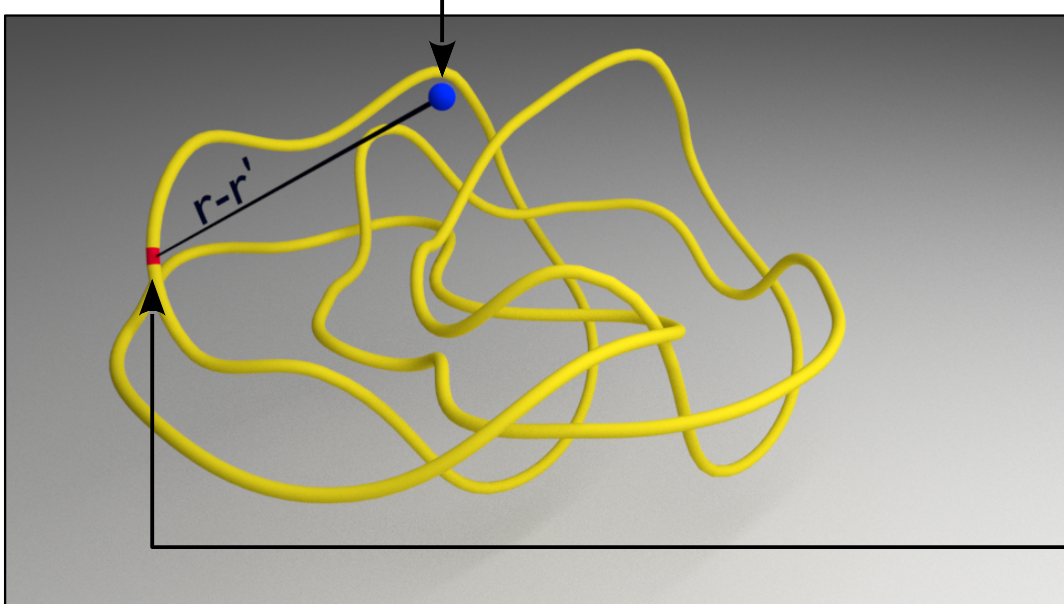
$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\nabla\times\int\frac{J(\vec{r}\,')}{|r-r\,'|}d^3r\,'=\nabla\times\boxed{\frac{\mu_0}{4\pi}\int\frac{J(\vec{r}\,')}{|r-r\,'|}d^3r\,'}$$

This is called
magnetic vector potential

$$\vec{B}(\vec{r})=\nabla\times\vec{A}(\vec{r}) \quad *$$

$\nabla\cdot(\nabla\times\vec{a})=0$ For an arbitrary ***A*** the magnetic induction ***B*** is divergenceless

*because $\nabla\times\nabla\phi=0$ one can add gradient of scalar function to ***A*** without changing ***B***.



observation point

$$\vec{B}(\vec{r})=\nabla\times\frac{\mu_0}{4\pi}\int\frac{J(\vec{r}\,')}{|r-r\,'|}d^3r\,'$$

Basic properties of static magnetic field

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{J(\vec{r}\,')}{|r-r'|} d^3 r' \qquad (1)$$

From (1), using the identity $\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$, the rotation of magnetic induction is:

$$\nabla \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \nabla \left(\nabla \cdot \int \frac{J(\vec{r}\,')}{|r-r'|} d^3 r' \right) - \boxed{\frac{\mu_0}{4\pi} \nabla^2 \left(\int \frac{J(\vec{r}\,')}{|r-r'|} d^3 r' \right)}$$

In the first term we use the identities $\nabla \cdot (\beta \vec{a}) = \vec{a} \cdot \nabla \beta + \beta \nabla \cdot \vec{a}$ and $\nabla \left(\frac{1}{|r-r'|} \right) = -\nabla' \left(\frac{1}{|r-r'|} \right)$ to get:

$$\nabla \cdot \int \frac{J(\vec{r}\,')}{|r-r'|} d^3 r' = \int \nabla \cdot \frac{J(\vec{r}\,')}{|r-r'|} d^3 r' = -\int \nabla' \cdot \frac{J(\vec{r}\,')}{|r-r'|} d^3 r' =$$

$$\boxed{\vec{a} = J(\vec{r}\,') \qquad \beta = \frac{1}{|r-r'|}}$$

$$-\int \left(J(\vec{r}\,') \cdot \nabla' \frac{1}{|r-r'|} + \frac{1}{|r-r'|} \nabla' \cdot J(\vec{r}\,') \right) d^3 r'$$

In manetostatics we assume $\nabla \cdot J(\vec{r}) = 0$ (**no charge accumulation**) so we get, remembering that nabla acts here on unprimed coordinates (for the second integral) [7]:

$$\nabla \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \nabla \left(\int J(\vec{r}\,') \cdot \nabla' \frac{1}{|r-r'|} d^3 r' \right) - \boxed{\frac{\mu_0}{4\pi} \int J(\vec{r}\,') \nabla^2 \frac{1}{|r-r'|} d^3 r'} \qquad (2)$$

MAGNETI RESIS

M. URBANIAK 2019

Basic properties of static magnetic field

$$\nabla\times\vec{B}(\vec{r})=\ldots-\frac{\mu_0}{4\pi}\int J(\vec{r}\,')\nabla^2\frac{1}{|r-r'|}d^3r' \qquad (2)$$

$$\nabla^2\left(\frac{1}{|r-r'|}\right)=-4\pi\delta(r-r')$$

$$\int_{-\infty}^{+\infty}f(x)\delta(x)dx=f(0)$$

$$\nabla\times\vec{B}(\vec{r})=-\frac{\mu_0}{4\pi}\nabla\left(\int J(\vec{r}\,')\cdot\nabla'\frac{1}{|r-r'|}d^3r'\right)+\mu_0 J(\vec{r})$$

We integrate the remaining integral using integration by parts: $\left(\frac{J(\vec{r}\,')}{|r-r'|}\right)'=J(\vec{r}\,')\left(\frac{1}{|r-r'|}\right)'+J'(\vec{r}\,')\left(\frac{1}{|r-r'|}\right)$

$$\int J(\vec{r}\,')\cdot\nabla'\frac{1}{|r-r'|}d^3r'=-\int\frac{\nabla'\cdot J(\vec{r}\,')}{|r-r'|}d^3r'+\int\nabla'\cdot\left(\frac{J(\vec{r}\,')}{|r-r'|}\right)d^3r'$$

The first integral vanishes by the divergence of current ($\nabla\cdot J(\vec{r})=0$). The second integral can be changed into a surface integral [8, 9] by applying:

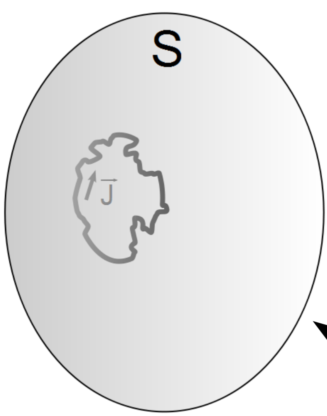
Gauss's theorem $\int_S\vec{A}\cdot dS=\int_V\nabla\cdot\vec{A}\,dV$

Basic properties of static magnetic field

Gauss's theorem $\int_S \vec{A}\cdot dS = \int_V \nabla\cdot\vec{A}\, dV$

$$\int \vec{J}(\vec{r}\,')\cdot\nabla'\frac{1}{|r-r'|}d^3r' = \int \nabla'\cdot\left(\frac{\vec{J}(\vec{r}\,')}{|r-r'|}\right)d^3r' = \int_S \frac{\vec{J}(\vec{r}\,')\cdot\vec{n}}{|r-r'|}dS$$

The integral vanishes as the volume enclosing currents is limited but the surface S can be placed **far away** from the currents. Finally we get:

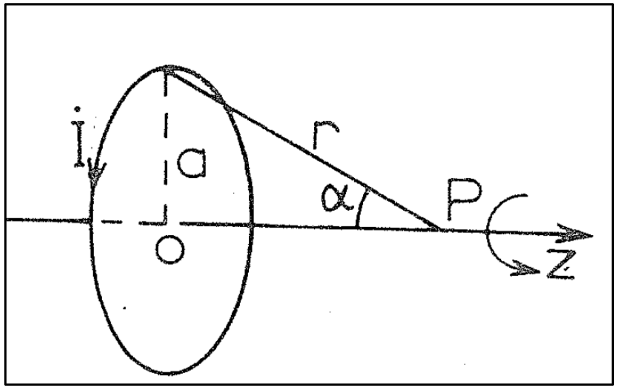


No current at surface S

$$\nabla\times\vec{B}(\vec{r}) = \mu_0\vec{J}(\vec{r})$$

- ***B*** is a solenoidal field

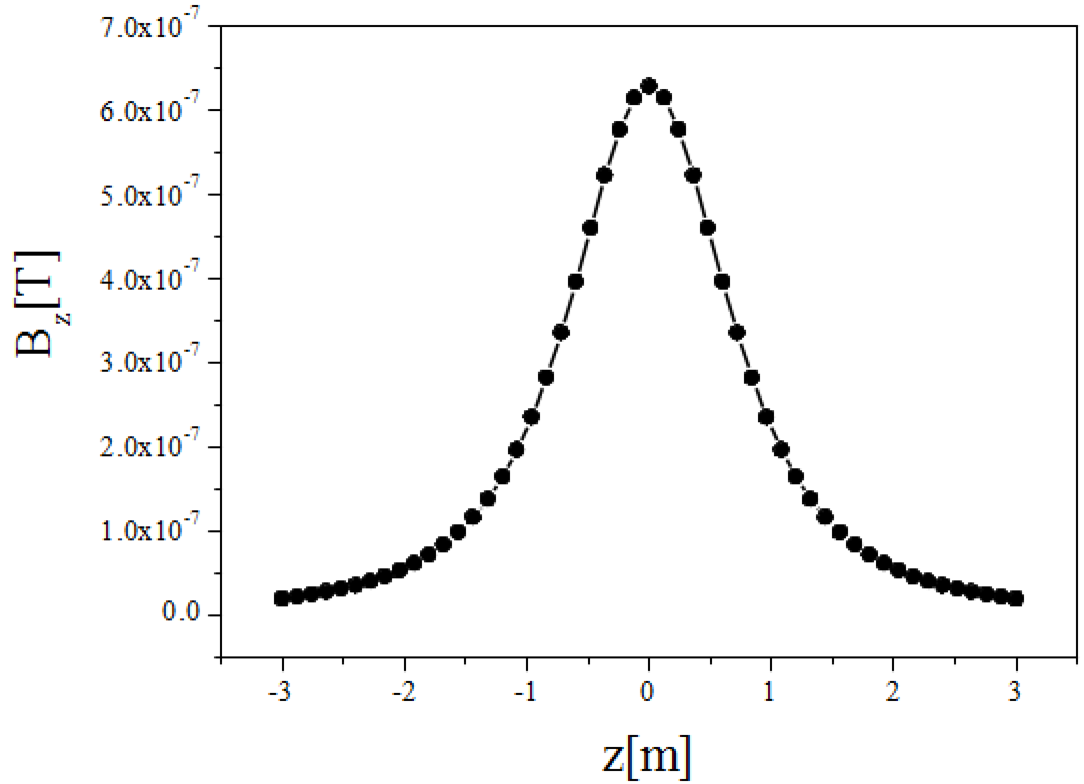
Magnetic field of circular currents loops



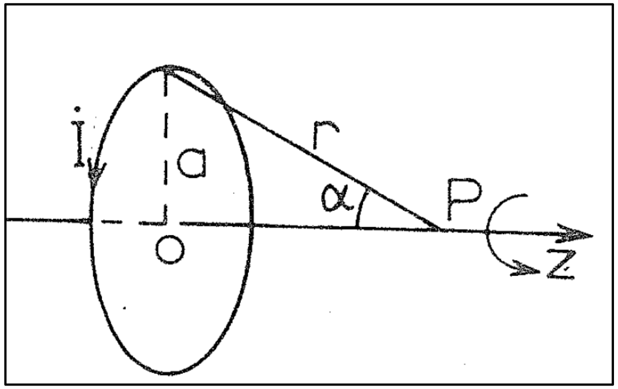
Source E. Durand [12]

- We are interested in the field produced by a current loop
- The exact formulas are quite difficult to derive [see 7, 12] (for off-axis positions)
- Here we do a numerical integration from Biot-Savart law (loop radius-1m, current 1A)

Field on symmetry axis



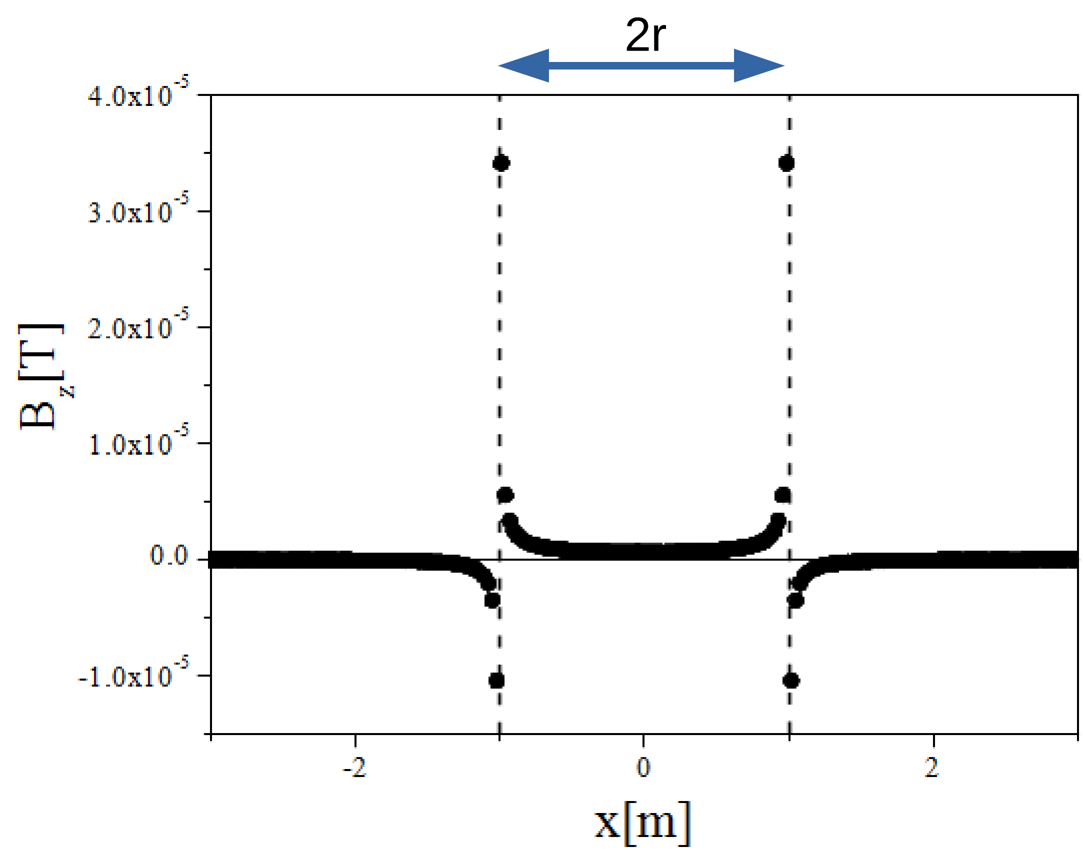
Magnetic field of circular currents loops



Source E. Durand [12]

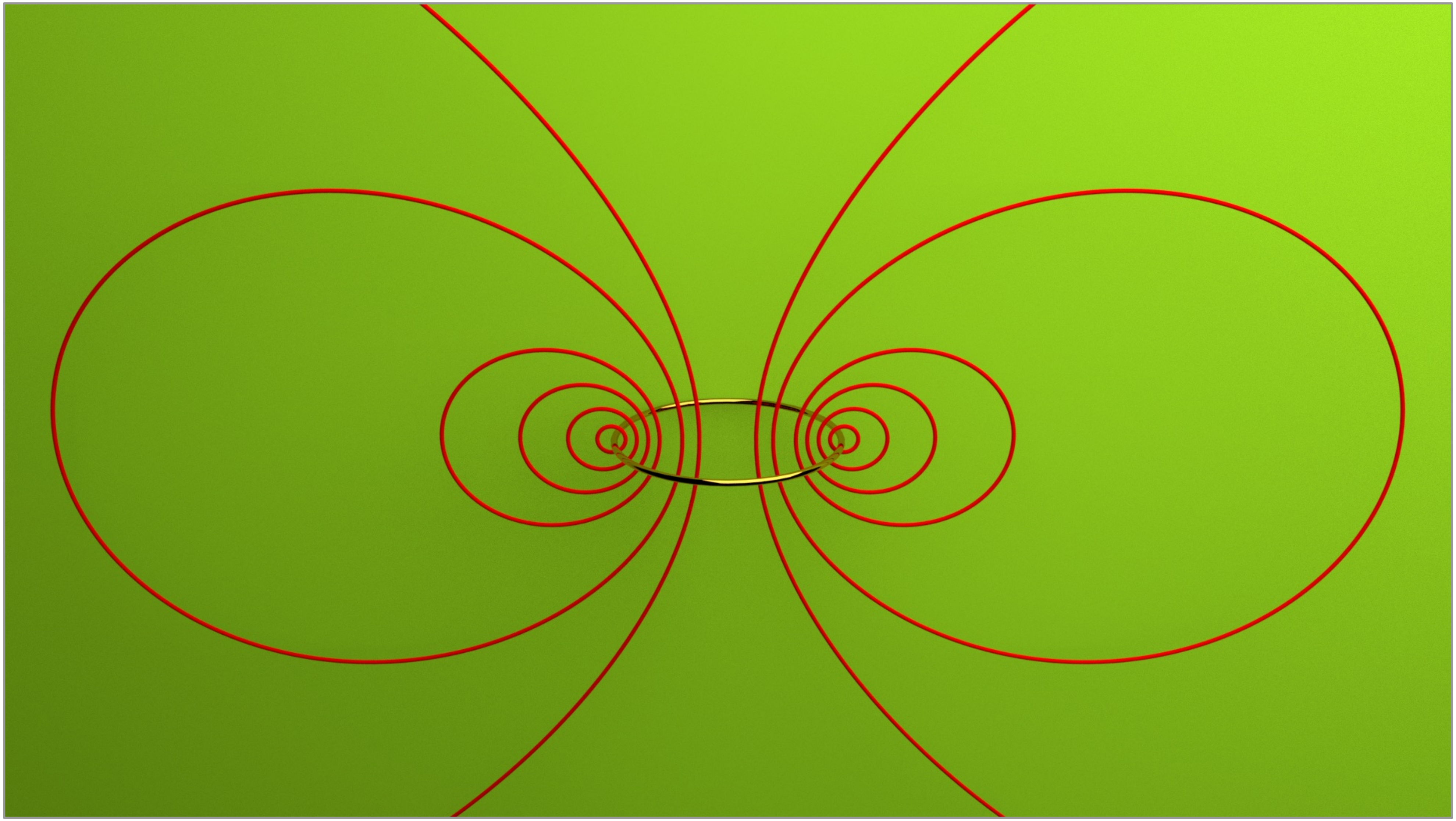
- We are interested in the field produced by a current loop
- The exact formulas are quite difficult to derive [see 7, 12] (for off-axis positions)
- Here we do a numerical integration from Biot-Savart law (loop radius-1m, current 1A)

Field on symmetry axis



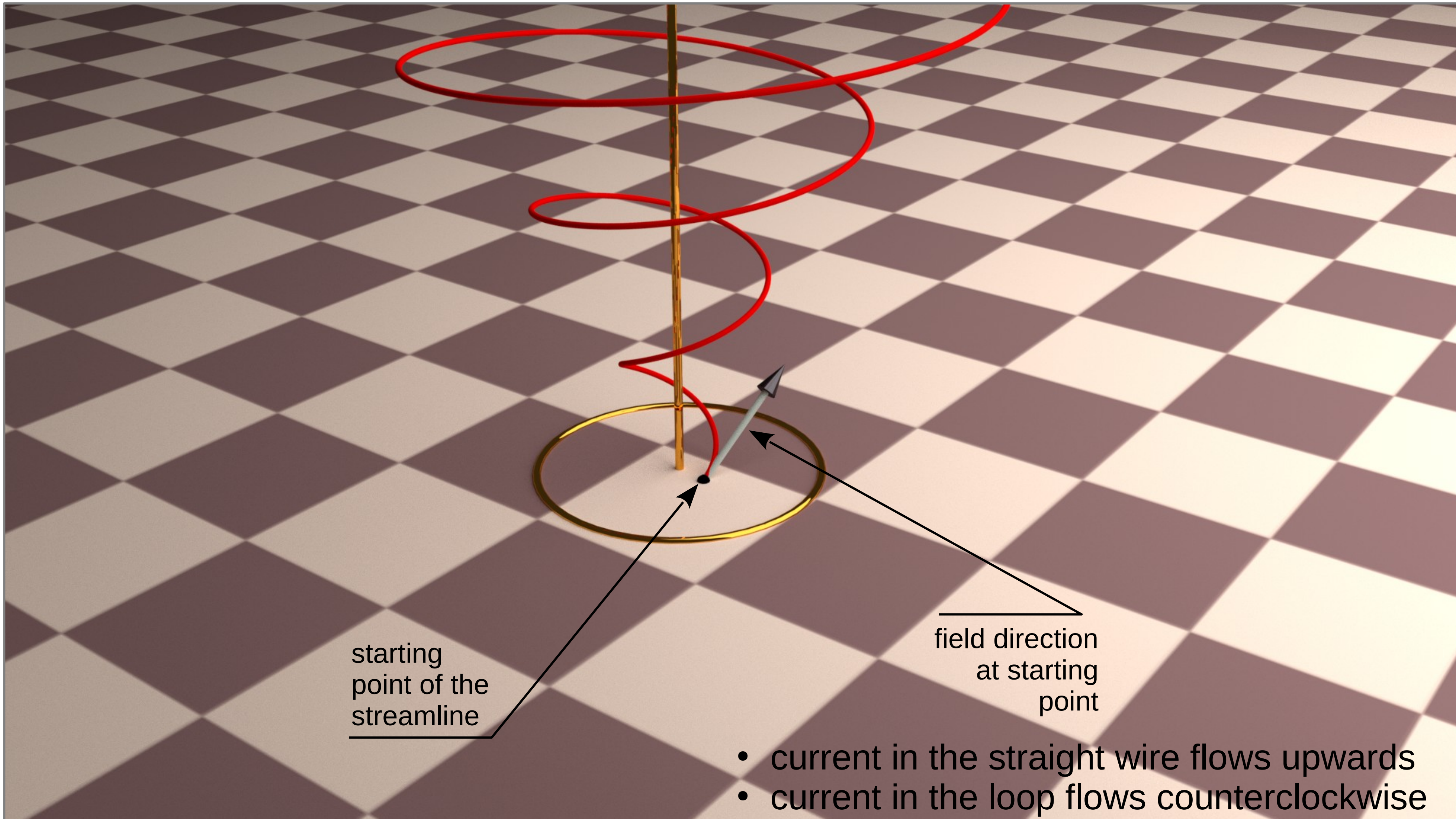
Magnetic field of circular currents loops

It is usual to display magnetic fields as streamlines:



Magnetic field streamlines of an infinite wire and a current carrying loop*

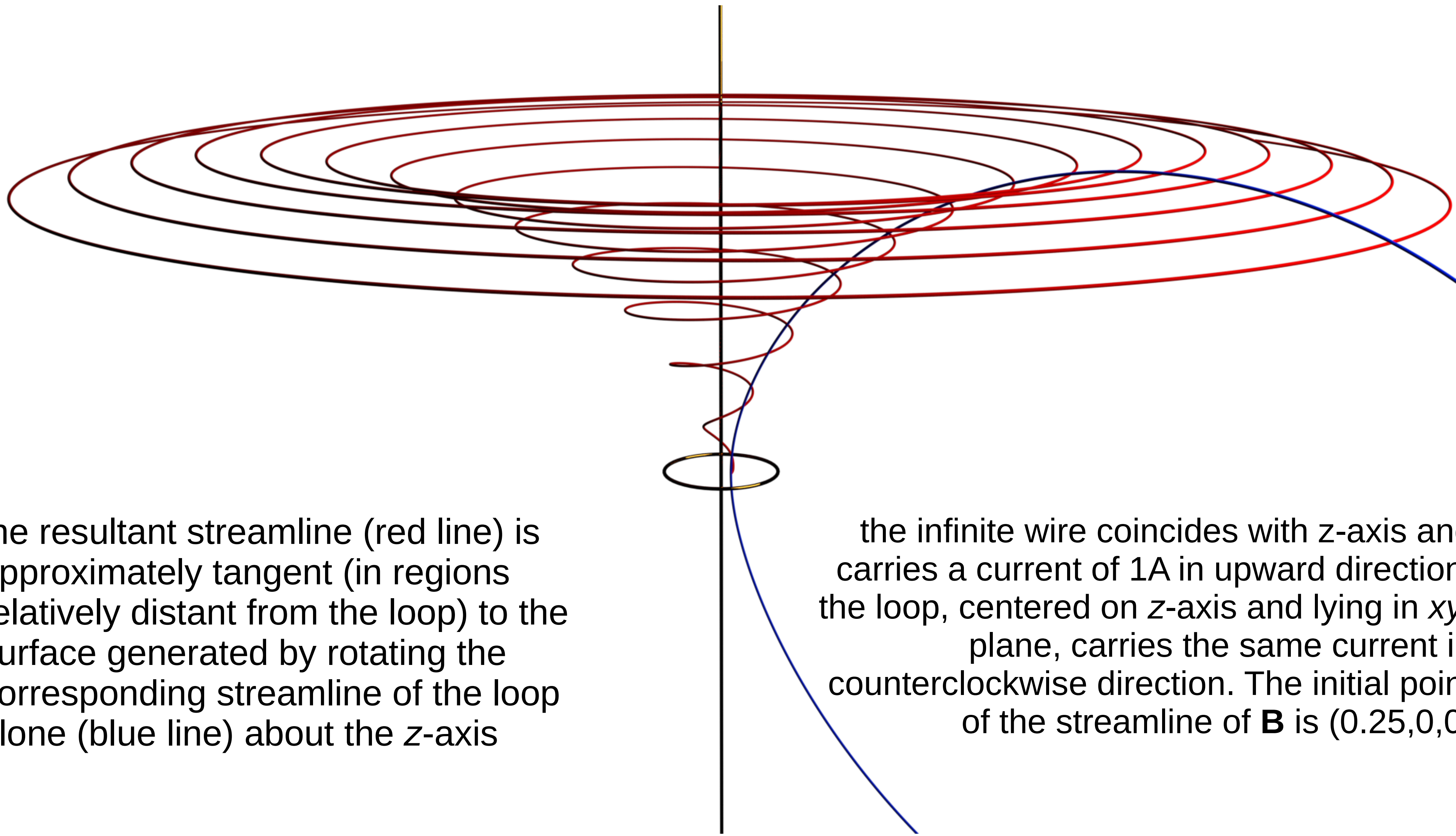
- problem initially considered by I. Y. Tamm (И. Е. Тамм) investigating tokamak (**то**роидальная **ка**мера с **ма**гнитными **к**атушками) device for controlling plasma with magnetic field [21]



This and the following streamlines were calculated using an algorithm of J. R. Pasta and S.M. Ulam with „displacement" of 10^{-3} r or 10^{-4} r (r – loop radius)– see Ref. 21. The loop field was calculated as a sum of Biot-Savart fields from 500 straight elements "constituting" the ring.

*E. Zilberti, IEEE Magn. Lett. **8**, 1306005 (2017)

Magnetic field streamlines of an infinite wire and a current carrying loop

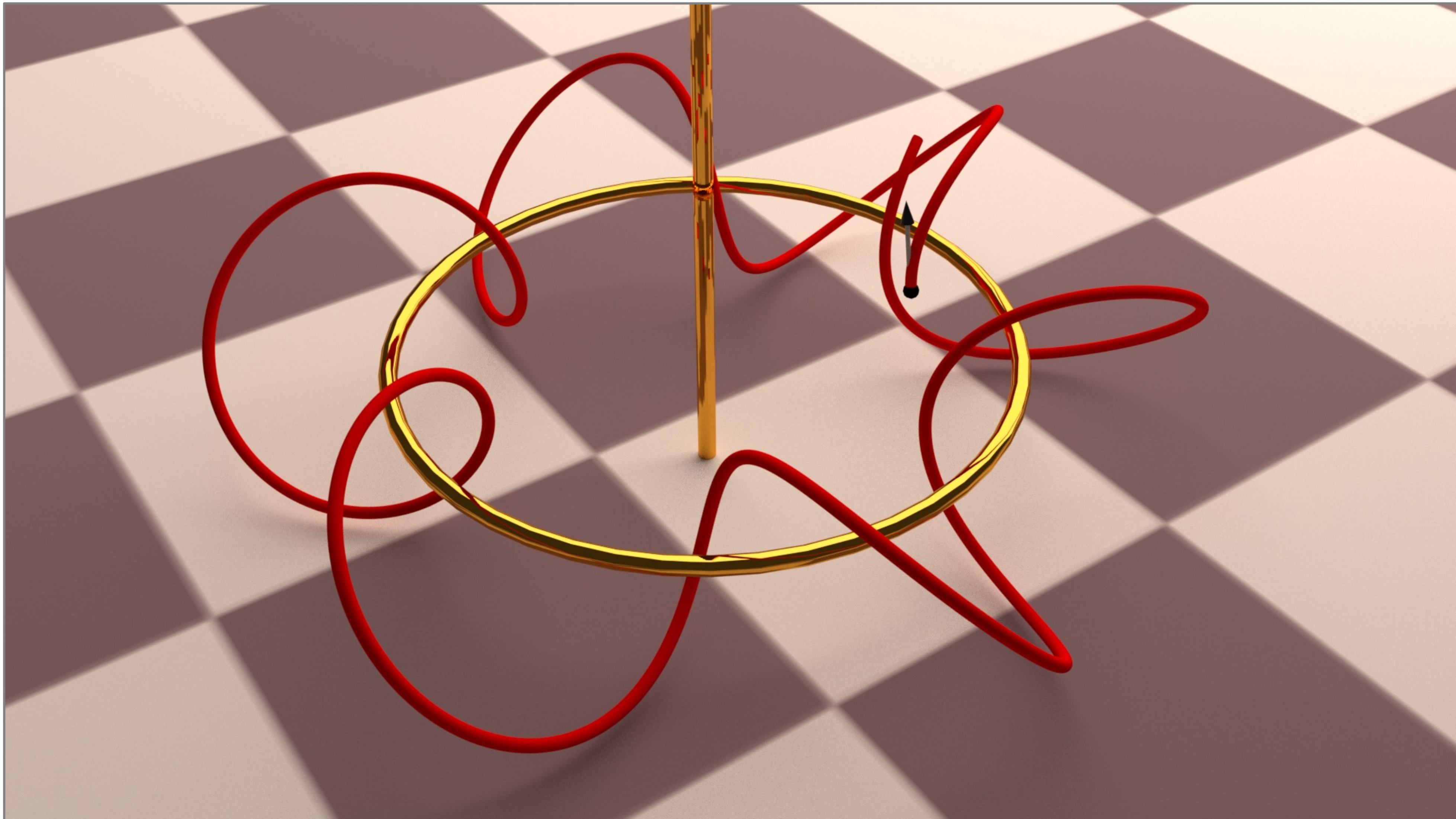


the resultant streamline (red line) is approximately tangent (in regions relatively distant from the loop) to the surface generated by rotating the corresponding streamline of the loop alone (blue line) about the *z*-axis

the infinite wire coincides with z-axis and carries a current of 1A in upward direction; the loop, centered on *z*-axis and lying in *xy*-plane, carries the same current in counterclockwise direction. The initial point of the streamline of **B** is (0.25,0,0)

Magnetic field streamlines of an infinite wire and a current carrying loop

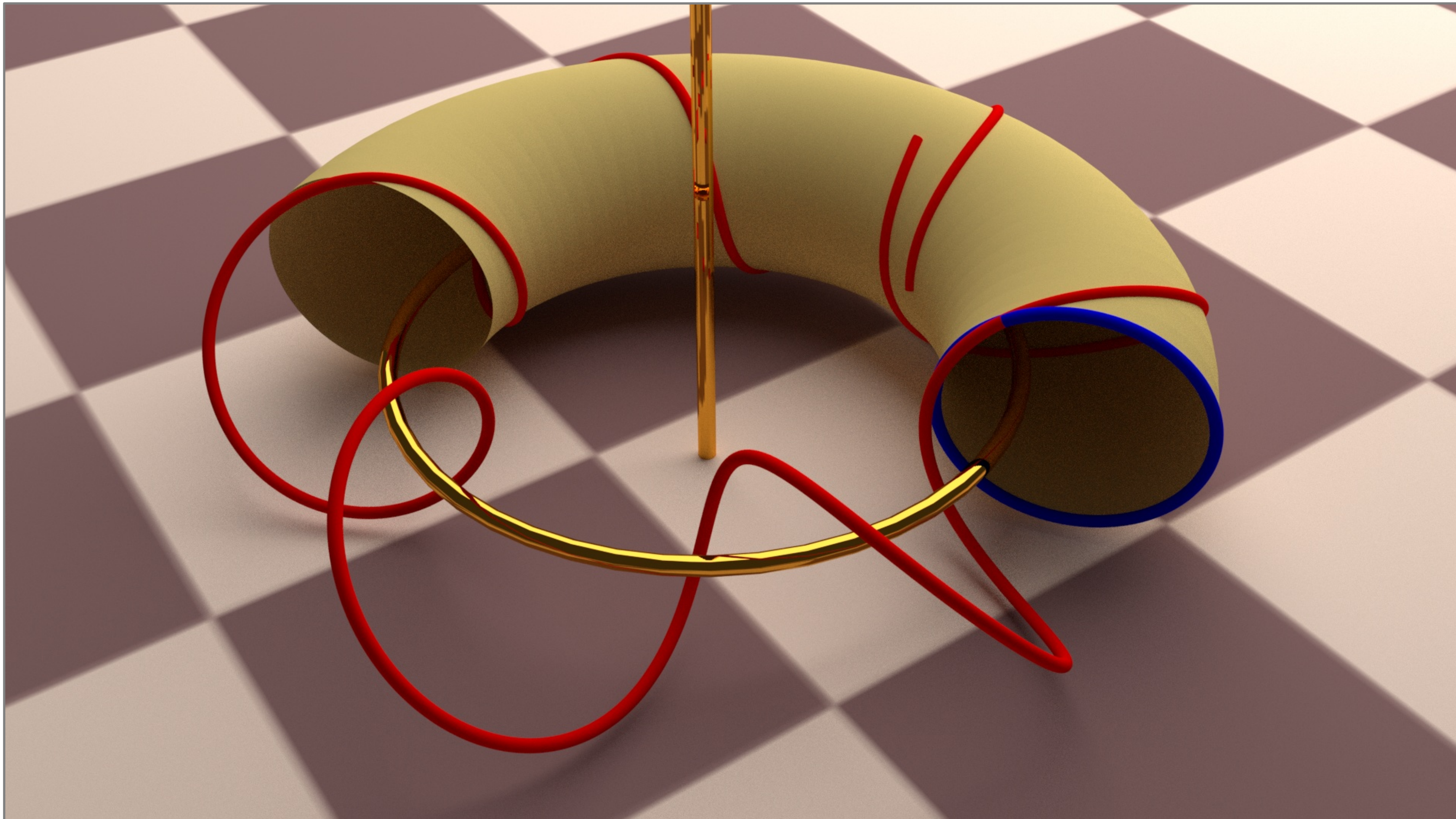
- the image shows streamlines for the same conductor/current arrangement as on the previous two slides but with the streamline starting at (0.75,0,0) point*
- note that the streamline is not closed



*the image corresponds approximately to Fig. 2 of E. Zilberti, IEEE Magn. Lett. **8**, 1306005 (2017)

Magnetic field streamlines of an infinite wire and a current carrying loop

- the image shows streamlines for the same conductor/current arrangement as on the previous two slides but with the streamline starting at (0.75,0,0) point P*
- the streamline winds over the surface created by rotating the streamline for the loop alone (blue line, starting at the same point P) around the *z*-axis (infinite wire)



*the image corresponds approximately to Fig. 2 of E. Zilberti, IEEE Magn. Lett. **8**, 1306005 (2017)

Magnetic field streamlines of an infinite wire and a current carrying loop

- it turns out that for this conductors arrangement the streamlines are closed only if the currents fulfill a relation [21]:

$$n_w I_L = k n_L I_w ,$$

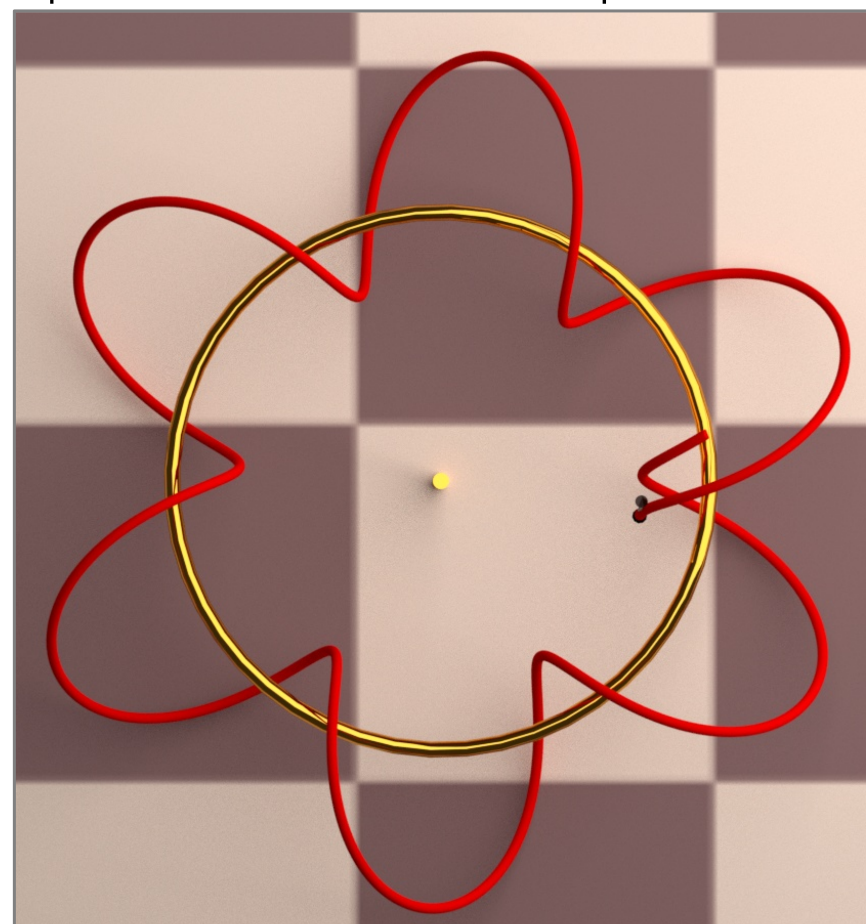
- where n_w and n_L are respectively integer numbers of rotations of a streamline around the wire and around the loop, and k is the integral depending on the geometry.

B-field lines are not always closed. Moreover, they may have a beginning or an end, in presence of singular points.

(L. Zilberti, [21])

An example of singular point: the center of a ,,Helmholtz coils" fed with opposite currents [21] or a special point on an axis of axially magnetized ring magnet (hollow cylinder*)

top view of the streamline from the previous slide:



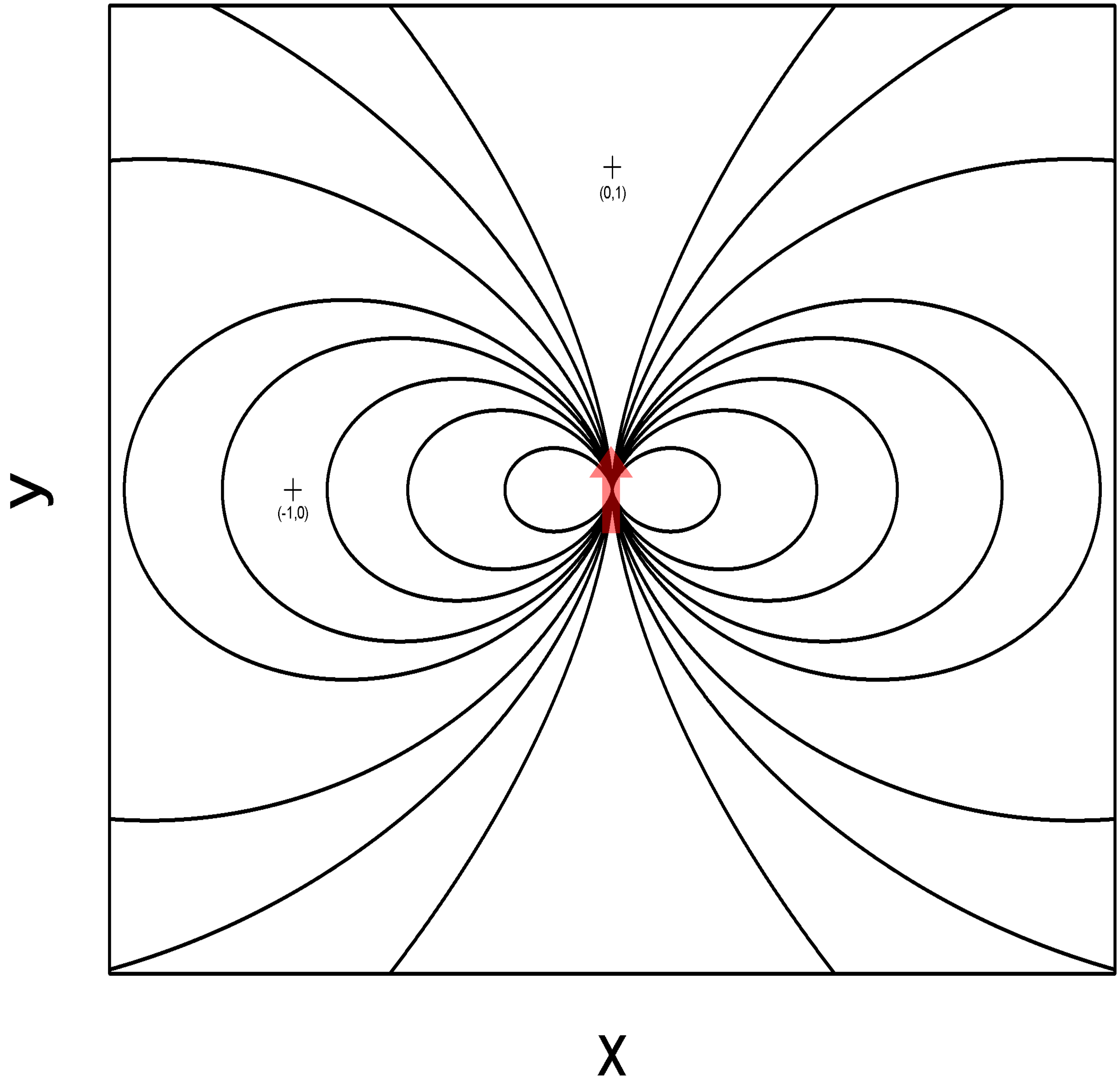
*see for example http://www.ifmpan.poznan.pl/~urbaniak/urbIFM20181023posemlowres.pdf

Magnetic field of a dipole

$$B_x = \frac{3m_z x z}{r^5}$$

$$B_y = \frac{3m_z y z}{r^5}$$

$$B_z = \frac{m_z (3z^2 - r^2)}{r^5}$$



MAGNETIC MATERIALS AND HYSTERESIS

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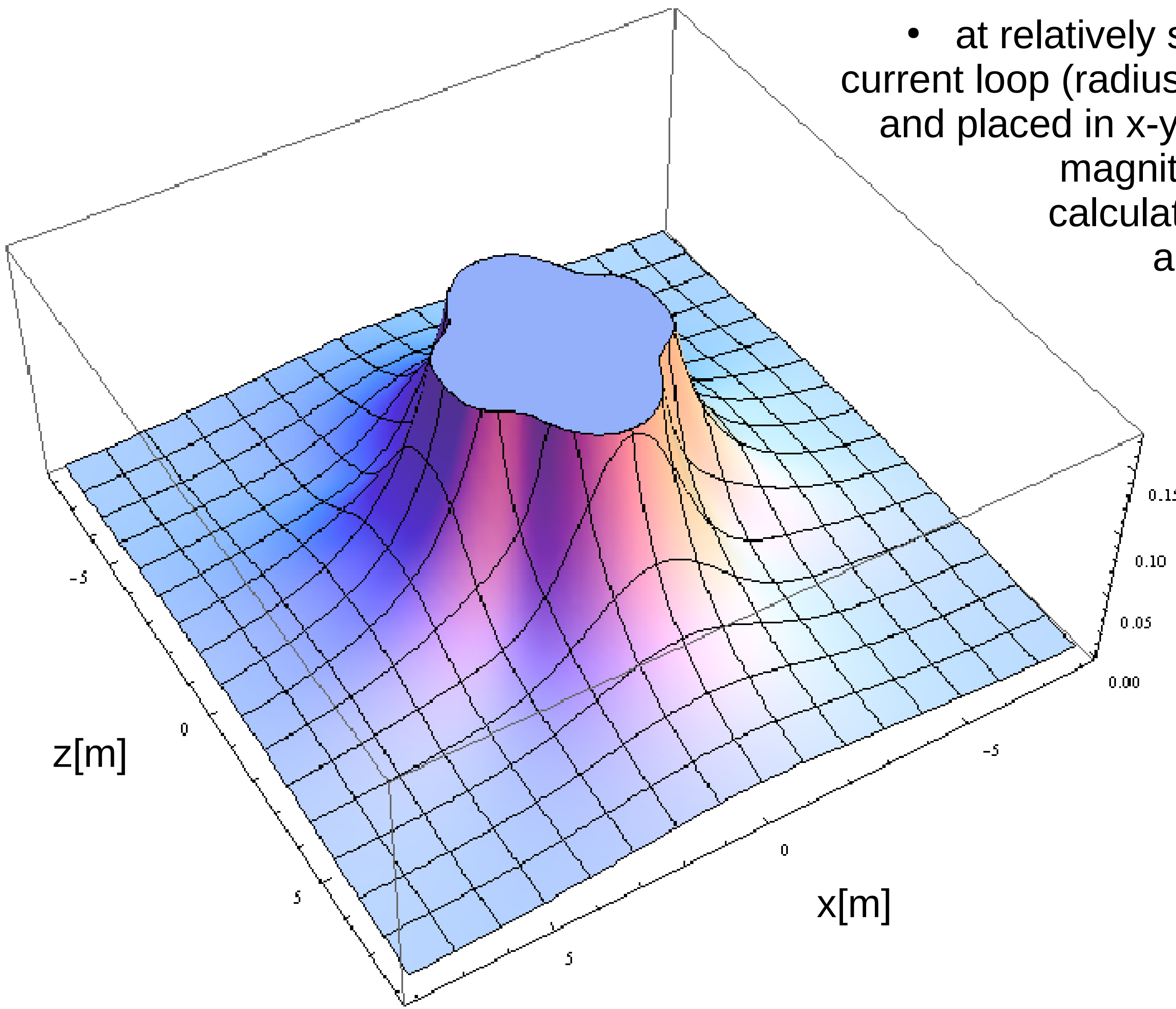
Magnetic field of a dipole – difference between dipole and current fields



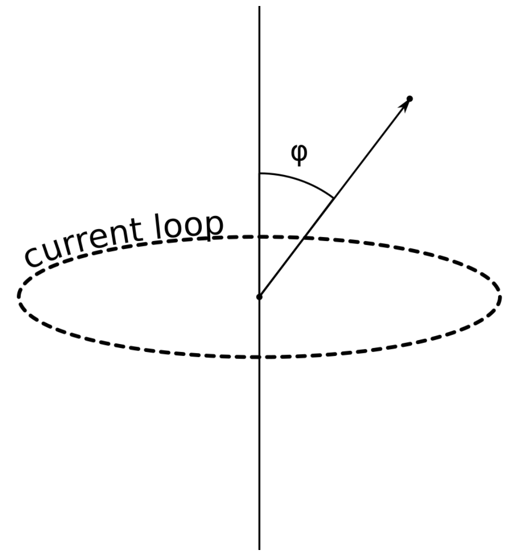
- at relatively small distances from the current loop (radius 1m, centered at (0,0,0) and placed in x-y plane) the difference of magnitudes between the fields calculated from Biot-Savart law and dipole approximation are well below 5%.

At large distances from the current distribution the field can be approximated by the dipole field.

Magnetic field of a dipole – difference between dipole and current fields

Far from the current loop the induction is given by the approximate expressions [7]:

$$B_r = \left(\frac{\mu_0 I r^2}{2}\right) \frac{\cos(\phi)}{r^3}$$



Comparing the above expressions with the dipole field: $p \propto I r^2$

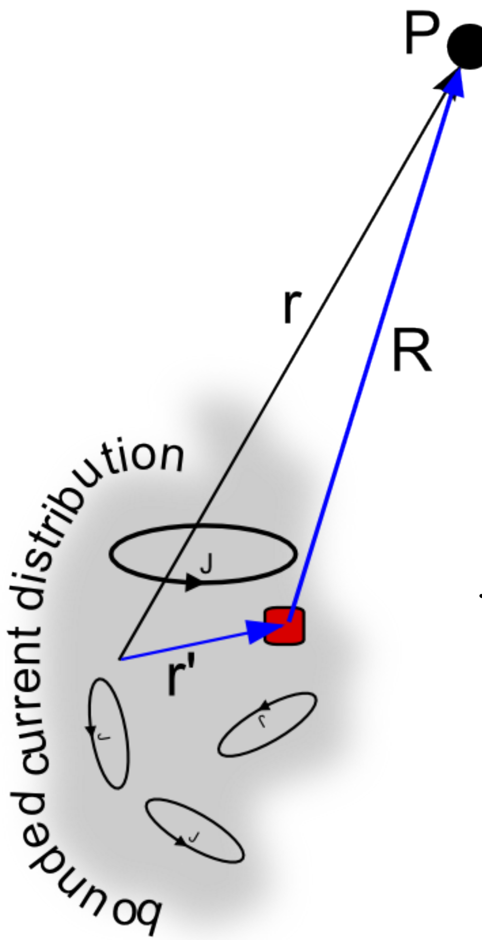
$$B_r = \frac{2 p \cos(\phi)}{r^3} \qquad B_\phi = \frac{p \sin(\phi)}{r^3}$$

We conclude:

Seen from distances large compared to the circular loop radius its magnetic induction ***B*** has a dipolar character.

Multipole expansion of magnetic fields

$$\boxed{\vec{B}(\vec{r}) = \nabla \times \frac{\mu_0}{4\pi} \int \frac{J(\vec{r}\,')}{|r-r'|} d^3 r'}$$



- We assume that the currents density is null outside some bounded volume
- The magnetic vector potential of the distribution is given by:

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3 r' \qquad (3)$$

- We express the denominator of the integrand in a Taylor series expansion* [9]:

$$f(\vec{r}-\vec{r}\,') = f(\vec{r}) - \left[x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right] f(\vec{r}) + \frac{1}{2} \sum_{i,j} x'_i x'_j \frac{\partial^2 f(\vec{r})}{\partial x'_i \partial x'_j} + \dots$$

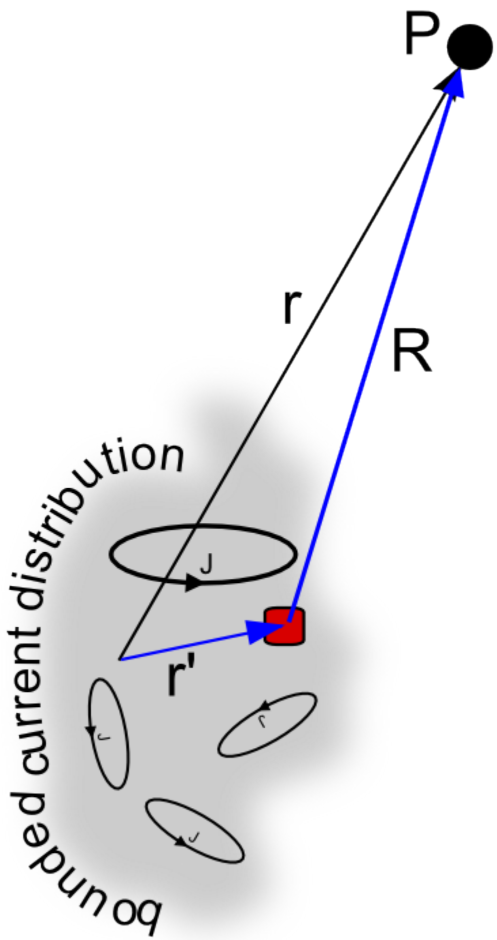
- We have:

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right) = \frac{-(x-x')}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}}$$

- It follows, taking derivatives at ***r***'=0, that:

$$\vec{r}\,' \cdot \nabla \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) = -\vec{r}\,' \cdot \frac{\vec{r}}{|\vec{r}|^3}$$

*usually one uses expansion into spherical harmonics



- At the moment we are interested in the first two terms of the expansion:

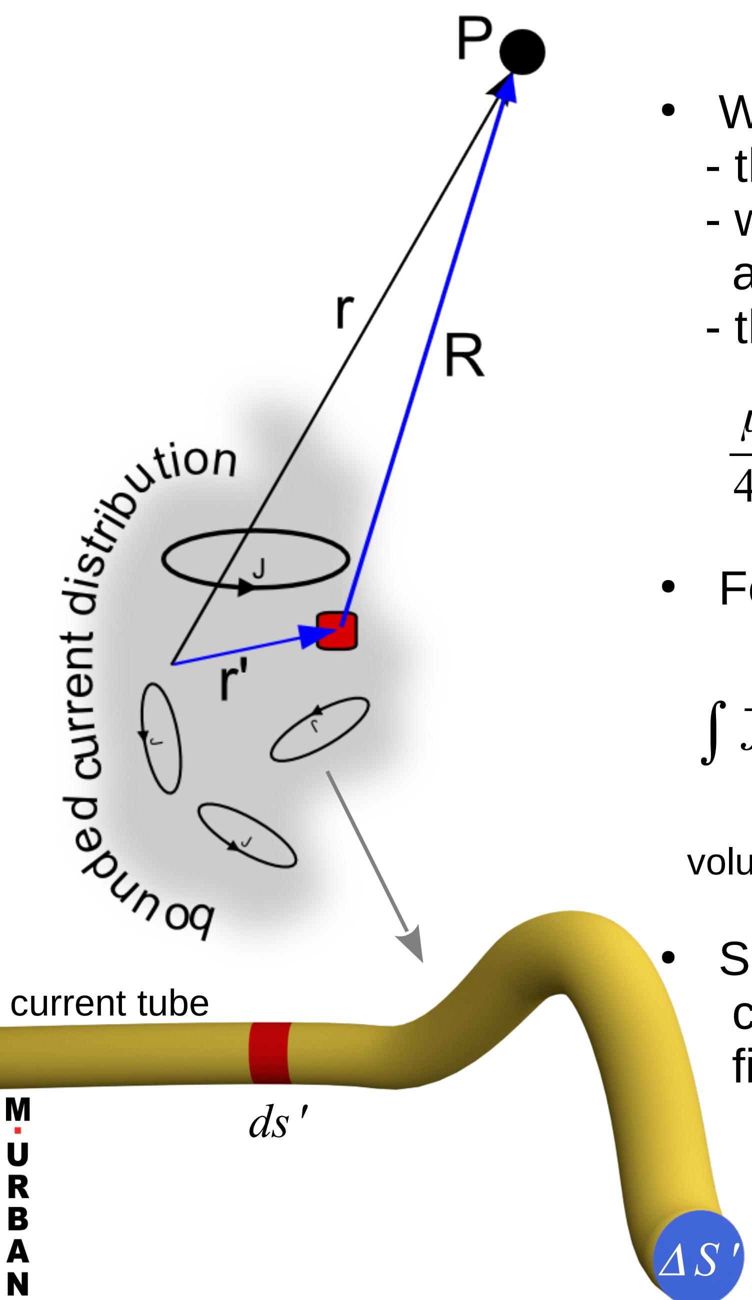
$$\frac{1}{|\vec{r}-\vec{r}\,'|}=\frac{1}{|\vec{r}|}+\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3}+\ldots$$

- Combining this with the expression (3) for vector potential we get:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\left(\frac{1}{|\vec{r}|}+\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3}+\ldots\right)d^3r'=$$
$$\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\left(\frac{1}{|\vec{r}|}\right)d^3r'+\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\left(\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3}\right)d^3r'+\ldots$$

Multipole expansion of magnetic fields

$$\boxed{\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\left(\frac{1}{|\vec{r}|}\right)d^3r'+\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\left(\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3}\right)d^3r'+\dots}$$



- We take on the first integral [9]:
  - the current distribution is a divergenceless
  - we can consider any time-independent current distribution as a sum of circulating currents
  - through each current tube there passes a current $I=\vec{J}\cdot\Delta S$

$$\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\left(\frac{1}{|\vec{r}|}\right)d^3r'=\frac{\mu_0}{4\pi}\left(\frac{1}{|\vec{r}|}\right)\int \vec{J}(\vec{r}\,')\,d^3r'$$

- For each current circuit we have:

$$\int \vec{J}(\vec{r}\,')\,dV'=\int \overbrace{\vec{J}(\vec{r}\,')\Delta S'}\cdot \underbrace{\Delta S'\cdot ds'}_{\text{volume element of the circuit}}=I\oint ds'$$

closed circuit

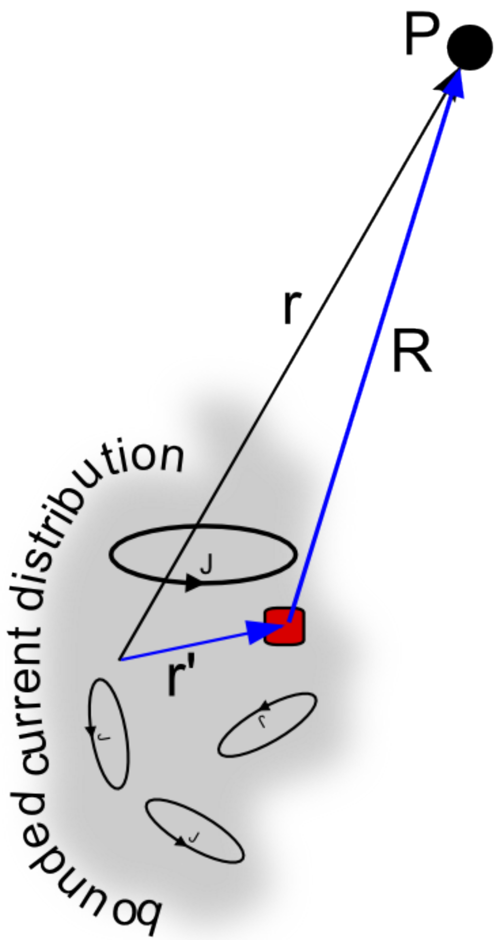
- Since path integral of ***ds*** along closed path is zero we conclude that the first term of the multipole expansion of the field of the current vanishes.

There are no magnetic monopoles *

*We have already seen that with $\nabla\cdot\vec{B}=0$.

Multipole expansion of magnetic fields

this slide shows an alternative to the derivation from the previous one



- Alternatively [14] the first integral* can be rewritten by the use of the vector identity:

$$\nabla\cdot(f\vec{A})=f\,\nabla\cdot\vec{A}+\vec{A}\cdot\nabla f$$

- It follows (as divergence of the current vanishes):

$$\nabla\cdot(x\vec{J})=x\underbrace{\nabla\cdot\vec{J}}_{0}+\vec{J}\cdot\nabla x=\vec{J}\cdot\nabla x=\vec{J}\,\hat{x}=J_x$$

- We have then:

$$\int J_x(\vec{r})\,d^3r=\int\nabla\cdot(x\vec{J})\,d^3r=\oiint x\vec{J}\,dS=0$$

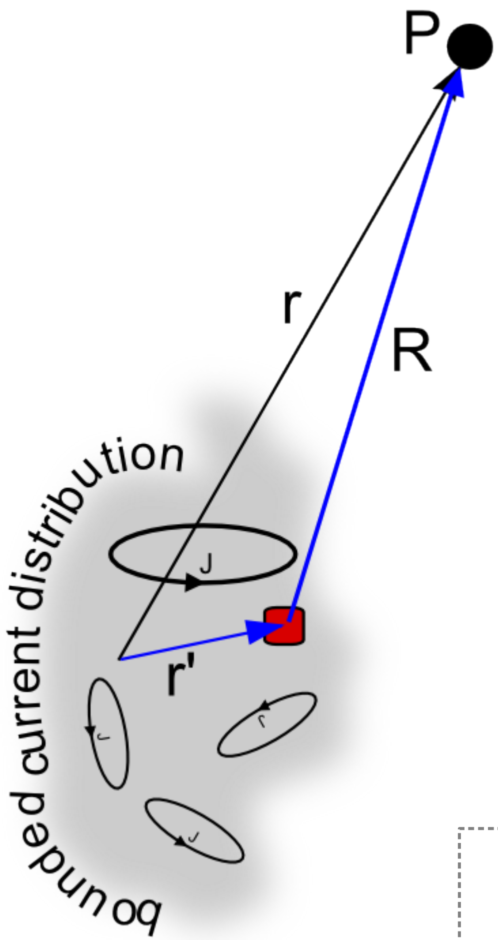
as the current density ***J*** vanishes at the outer boundary.

- Similar consideration holds for other Cartesian components of ***J***, so finally we have:

$$\frac{\mu_0}{4\pi}\left(\frac{1}{|\vec{r}|}\right)\int\vec{J}(\vec{r}\,')\,d^3r'=0$$

* $\boxed{\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\left(\frac{1}{|\vec{r}|}\right)d^3r'+\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\left(\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3}\right)d^3r'+\ldots}$

Multipole expansion of magnetic fields



- We rewrite now the second integral of Taylor expansion:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\left(\frac{1}{|\vec{r}|}\right)d^3r' + \boxed{\frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}\,')\left(\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3}\right)d^3r'} + \ldots$$

- We have, for arbitrary scalar functions *f* and *g*:

$$\frac{\partial}{\partial x}(gf\vec{J})=fg\frac{\partial}{\partial x}\vec{J}+\vec{J}\frac{\partial}{\partial x}fg=fg\frac{\partial}{\partial x}\vec{J}+f\vec{J}\frac{\partial}{\partial x}g+g\vec{J}\frac{\partial}{\partial x}f$$

$$g\vec{J}\frac{\partial}{\partial x}f=\frac{\partial}{\partial x}(gf\vec{J})-fg\frac{\partial}{\partial x}\vec{J}-f\vec{J}\frac{\partial}{\partial x}g$$

- Going now 3D [15]:

three cartesian coordinates

$$\int g\vec{J}\cdot\nabla' f\, d^3x' = \sum_i \int g J_i \partial'_i f\, d^3x' =$$

$$\sum_i \int \left[\partial'_i(J_i f g) - f J_i \partial'_i g - f g \partial'_i J_i\right] d^3x' =$$

$$\sum_i \partial'_i J_i = \text{divergence}$$

$$\int \left[\nabla'\cdot(\vec{J} f g) - f\vec{J}\cdot\nabla' g - f g \nabla'\cdot\vec{J}\right] d^3x' =$$

$$\oiint f g \vec{J}\cdot dS - \int \left[f\vec{J}\cdot\nabla' g + f g \nabla'\cdot\vec{J}\right] d^3x'$$

= 0 since current density vanishes on the outer boundary

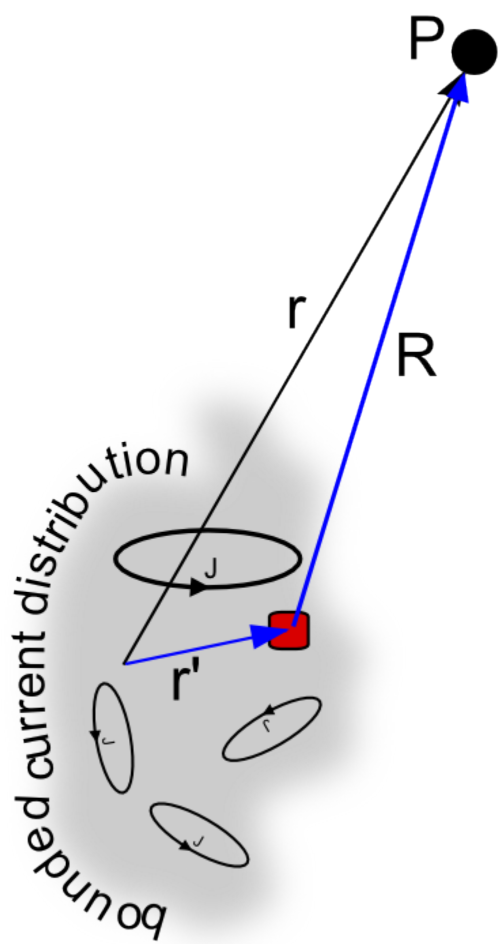
- Rewriting yields:

$$\int \left(g\vec{J}\cdot\nabla' f + f\vec{J}\cdot\nabla' g + f g\nabla'\cdot\vec{J}\right) d^3x' = 0 \qquad (4)$$

Gauss's theorem

$$\int_S \vec{A}\cdot dS = \int_V \nabla\cdot\vec{A}\, dV$$

Multipole expansion of magnetic fields



- We rewrite (4) using the substitutions $f = x'_i \qquad g = x'_j$

$$\int \left(g \vec{J}\cdot\nabla' f + f \vec{J}\cdot\nabla' g + f\, g\, \underbrace{\nabla'\cdot\vec{J}}_{0} \right) d^3x' = 0 \tag{4}$$

$$\int \left(x'_j \vec{J}\cdot\nabla' x'_i + x'_i \vec{J}\cdot\nabla' x'_j \right) d^3x' = 0$$

$$\int \left(x'_j \vec{J}\cdot\hat{x}_i + x'_i \vec{J}\cdot\hat{x}_j \right) d^3x' = 0$$

$$\int \left(x'_j J_i + x'_i J_j \right) d^3x' = 0$$

- We note that:

$$\int \left(x'_j J_i \right) d^3x' = \int \left(-x'_i J_j \right) d^3x'$$

$$\int 2x'_j J_i d^3x' = \int \left(x'_j J_i + x'_j J_i \right) d^3x' = \int \left(x'_j J_i - x'_i J_j \right) d^3x'$$

$$\int \left(x'_j J_i \right) d^3x' = -\frac{1}{2} \int \left(x'_i J_j - x'_j J_i \right) d^3x' \tag{5}$$

- We calculate now *i-th* **component** of the second term of the expansion of vector potential ***A***:

$$\boxed{A_i(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{r}}{|\vec{r}|^3} \cdot \int J_i(\vec{r}\,')\vec{r}\,' d^3r'}$$

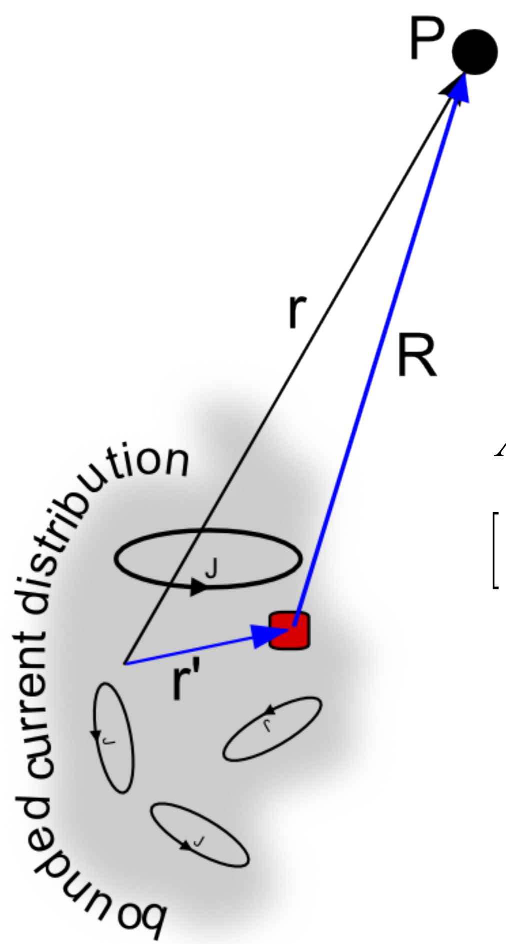
scalar

does not depend on r' - can be put outside the integral

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \left(\frac{1}{|\vec{r}|} \right) d^3r' + \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r}\,') \left(\frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^3} \right) d^3r' + \dots}$$

Multipole expansion of magnetic fields

$$\int (x'_j J_i)\, d^3x' = -\frac{1}{2}\int (x'_i J_j - x'_j J_i)\, d^3x' \quad (5)$$



- We rewrite the expression for the component *i* of potential ***A***:

$$\boxed{A_i(\vec r) = \frac{\mu_0}{4\pi}\frac{\vec r}{|\vec r|^3}\cdot\int J_i(\vec r\,')\,\vec r\,'\, d^3r'} = \frac{\mu_0}{4\pi}\frac{1}{|\vec r|^3}\sum_j r_j \int r'_j J_i(\vec r\,')\, d^3r' =$$

$$-\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec r|^3}\sum_j r_j \int (r'_i J_j - r'_j J_i)\, d^3r'$$

using (5)

$r_i \equiv x_i$

- The *x* component of ***A*** is then:

$$A_x(\vec r) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec r|^3}\sum_j r_j \int (r'_x J_j - r'_j J_x)\, d^3r' = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec r|^3}\times$$

$$\left[\underbrace{r_x \int (r'_x J_x - r'_x J_x)\, d^3r'}_{0} + r_y \int (\boxed{r'_x J_y} - \boxed{r'_y J_x})\, d^3r' + r_z \int (r'_x J_z - r'_z J_x)\, d^3r'\right]$$

etc.

- Note that: $[\vec r \times (\vec r\,' \times \vec J)]_x = r_y r'_x J_y - r_y r'_y J_x - r_z r'_z J_x + r_z r'_x J_z$

one component

and consequently:

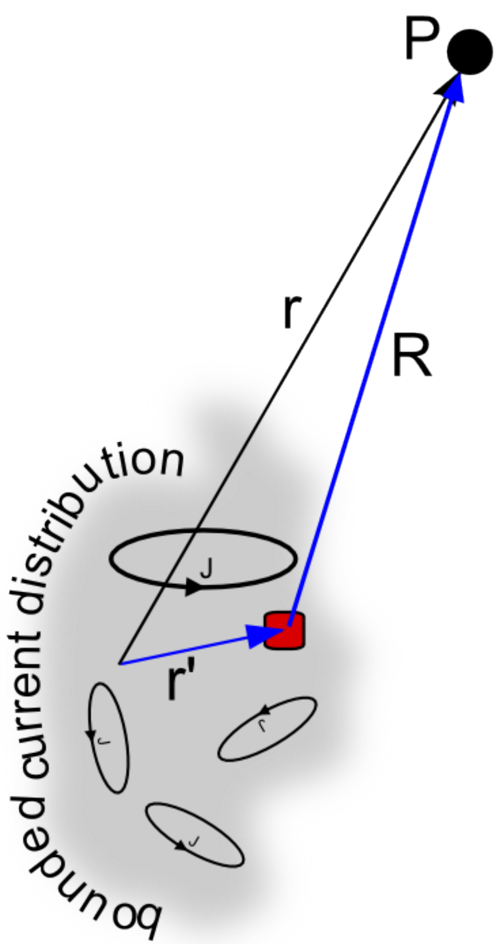
$$A_x(\vec r) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec r|^3}\left[\vec r \times \int (\vec r\,' \times \vec J(\vec r\,'))\, d^3r'\right]_x$$

$$\boxed{\vec A(\vec r) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec r|^3}\,\vec r \times \int (\vec r\,' \times \vec J(\vec r\,'))\, d^3r'}$$

Vector potential from the first two terms of the expansion:

$$\frac{1}{|\vec r - \vec r\,'|} = \frac{1}{|\vec r|} + \frac{\vec r \cdot \vec r\,'}{|\vec r|^3} + \ldots$$

Multipole expansion of magnetic fields



$$\vec{A}(\vec{r}) = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|^3}\vec{r}\times\boxed{\int(\vec{r}\,'\times\vec{J}(\vec{r}\,'))d^3r\,'}$$

- We define **magnetic dipole moment** of current distribution [7]:

$$\vec{m} = \frac{1}{2}\int(\vec{r}\,'\times\vec{J}(\vec{r}\,'))d^3r\,' \qquad \left[m\cdot\frac{A}{m^2}\cdot m^3 = A\cdot m^2\right]$$

- The integrand of the above expression is called **magnetization**

$$\vec{M}(\vec{r}) = \frac{1}{2}\vec{r}\,'\times\vec{J}(\vec{r}\,')$$

$M_{Fe} \approx 1.7\times10^6$ A/m

$M_{Co} \approx 1.4\times10^6$ A/m

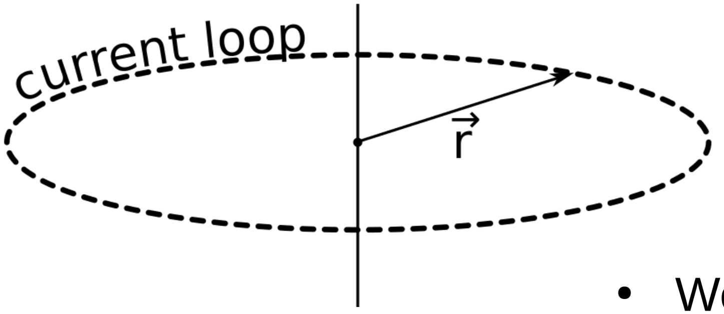
$M_{Ni} \approx 0.5\times10^6$ A/m

at RT

Note that in ferromagnetic materials magnetic magnetization is due mainly to spin, i.e., the property of electron independent of current flow*

*electron orbiting a nucleous can be thought of as a current flowing in a circle

Multipole expansion of magnetic fields



$$|M| = \frac{1}{2} r J$$

$V = 2\pi r$ - length as a volume

$$m = M V = \pi r^2 J$$

$$m[A \cdot m^2]$$

- We define **magnetic dipole moment** of current distribution [7]:

$$\vec{m} = \frac{1}{2}\int (\vec{r}\,' \times \vec{J}(\vec{r}\,'))\, d^3 r\,' \qquad \left[m \cdot \frac{A}{m^2} \cdot m^3 = A \cdot m^2\right]$$

- The integrand of the above expression is called **magnetization**

$$\vec{M}(\vec{r}\,') = \frac{1}{2}\vec{r}\,' \times \vec{J}(\vec{r}\,')$$

$M_{Fe} \approx 1.7 \times 10^6$ A/m

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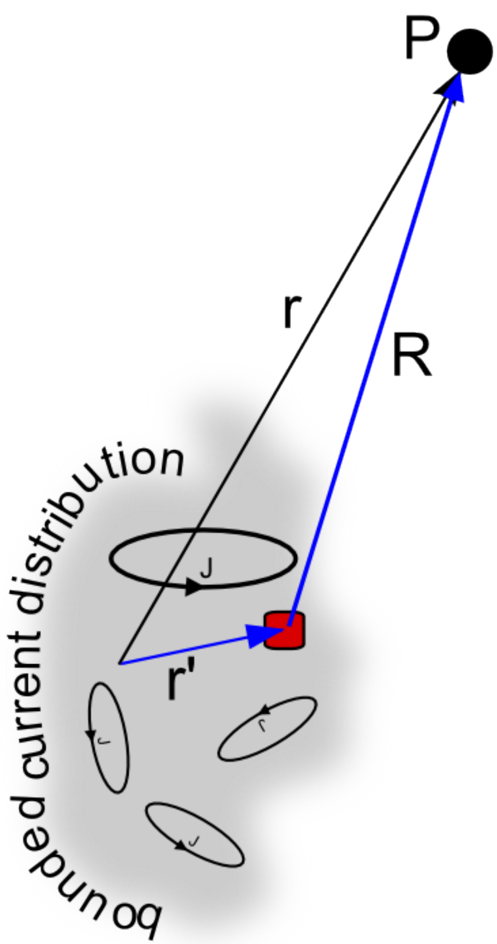
at RT

Note that in ferromagnetic materials magnetic magnetization is due mainly to spin, i.e., the property of electron independent of current flow*

Multipole expansion of magnetic fields – dipole approximation



- From the expression for the potential ***A*** and the definition of ***m*** we have [14]:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|^3}\vec{m}\times\vec{r}$$

- We have from the definition of $\vec{A}$: $\vec{B}(\vec{r})=\nabla\times\vec{A}(\vec{r})$

- Using $\nabla\times(\vec{a}\,f)=f\,\nabla\times\vec{a}-\vec{a}\times(\nabla f)$ we obtain:

$\vec{a}=\vec{m}\times\vec{r}$

$f=\frac{1}{|\vec{r}|^3}$

$$\vec{B}(\vec{r})=\nabla\times\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{r}|^3}\nabla\times(\vec{m}\times\vec{r})-(\vec{m}\times\vec{r})\times\nabla\frac{1}{|\vec{r}|^3}\right]$$

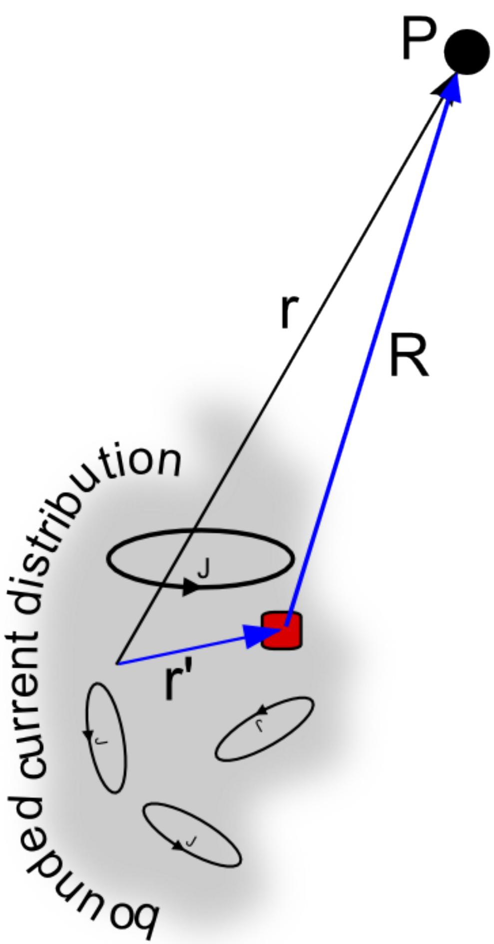
- Using $\nabla\times(\vec{a}\times\vec{b})=(\vec{b}\cdot\nabla)\vec{a}-(\vec{a}\cdot\nabla)\vec{b}+\vec{a}(\nabla\cdot\vec{b})-\vec{b}(\nabla\cdot\vec{a})$ we obtain:

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}\left[(\vec{r}\cdot\nabla)\vec{m}-(\vec{m}\cdot\nabla)\vec{r}+\vec{m}(\nabla\cdot\vec{r})-\vec{r}(\nabla\cdot\vec{m})\right]-(\vec{m}\times\vec{r})\times\nabla\frac{1}{|\vec{r}|^3}\right.$$

0 as ***m*** does not depend on ***r***

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}\left[-(\vec{m}\cdot\nabla)\vec{r}+\vec{m}(\nabla\cdot\vec{r})\right]-(\vec{m}\times\vec{r})\times\nabla\frac{1}{|\vec{r}|^3}\right\}$$

Multipole expansion of magnetic fields



- We have, from the definition of nabla:

$$(\vec{m}\cdot\nabla)\vec{r}=\vec{m}$$

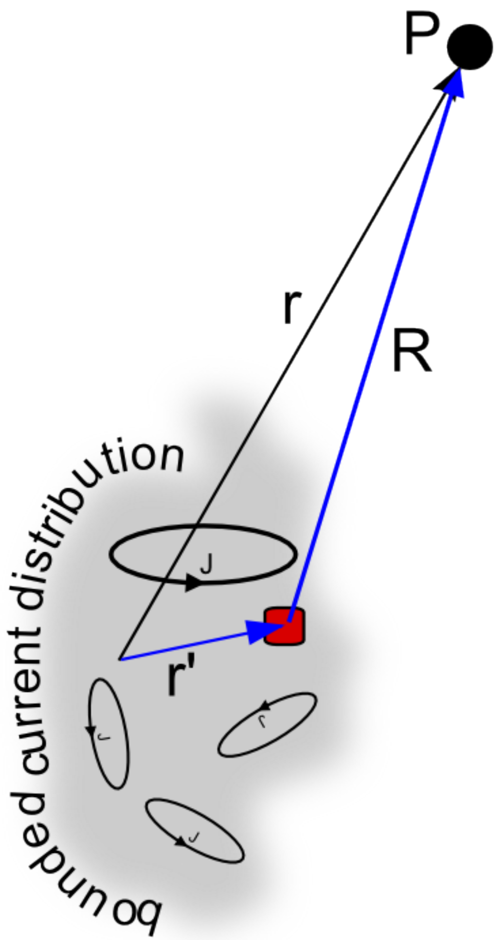
$$\nabla\cdot\vec{r}=3$$

$$\nabla\left(\frac{1}{|\vec{r}|^3}\right)=-\frac{3\,\vec{r}}{|\vec{r}|^5}$$

$$\vec{r}=x_i\hat{x}_i$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}\left[-(\vec{m}\cdot\nabla)\vec{r}+\vec{m}(\nabla\cdot\vec{r})\right]-(\vec{m}\times\vec{r})\times\nabla\frac{1}{|\vec{r}|^3}\right\}$$

Multipole expansion of magnetic fields



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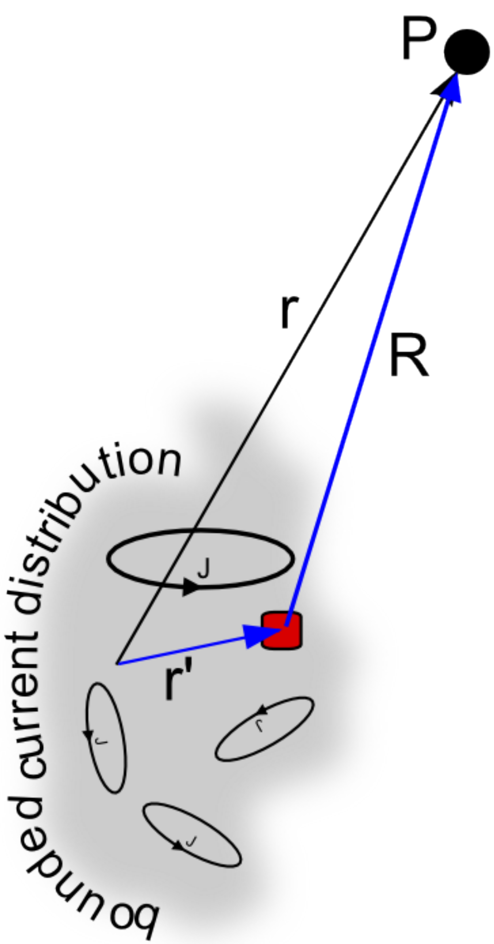
$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}\left[-(\vec{m}\cdot\nabla)\vec{r}+\vec{m}(\nabla\cdot\vec{r})\right]-(\vec{m}\times\vec{r})\times\nabla\frac{1}{|\vec{r}|^3}\right\}$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}\left[-\vec{m}+3\,\vec{m}\right]+\frac{3}{|\vec{r}|^5}(\vec{m}\times\vec{r})\times\vec{r}\right\}$$

Multipole expansion of magnetic fields



- To transform the third term we use the identity:

$$(\vec{b}\times\vec{c})\times\vec{a}=\vec{c}(\vec{a}\cdot\vec{b})-\vec{b}(\vec{a}\cdot\vec{c})$$

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}\left[-\vec{m}+3\vec{m}\right]+\frac{3}{|\vec{r}|^5}(\vec{m}\times\vec{r})\times\vec{r}\right\}=$$

$$\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}2\vec{m}+\frac{3}{|\vec{r}|^5}\left(\vec{r}(\vec{r}\cdot\vec{m})-\vec{m}(\vec{r}\cdot\vec{r})\right)\right\}=$$

$$\frac{\mu_0}{4\pi}\left\{\frac{1}{|\vec{r}|^3}2\vec{m}+\frac{3}{|\vec{r}|^5}\left(\vec{r}(\vec{m}\cdot\vec{r})-\vec{m}|\vec{r}|^2\right)\right\}=\frac{\mu_0}{4\pi}\left\{\frac{-\vec{m}}{|\vec{r}|^3}+\frac{3}{|\vec{r}|^5}\left(\vec{r}(\vec{m}\cdot\vec{r})\right)\right\}$$

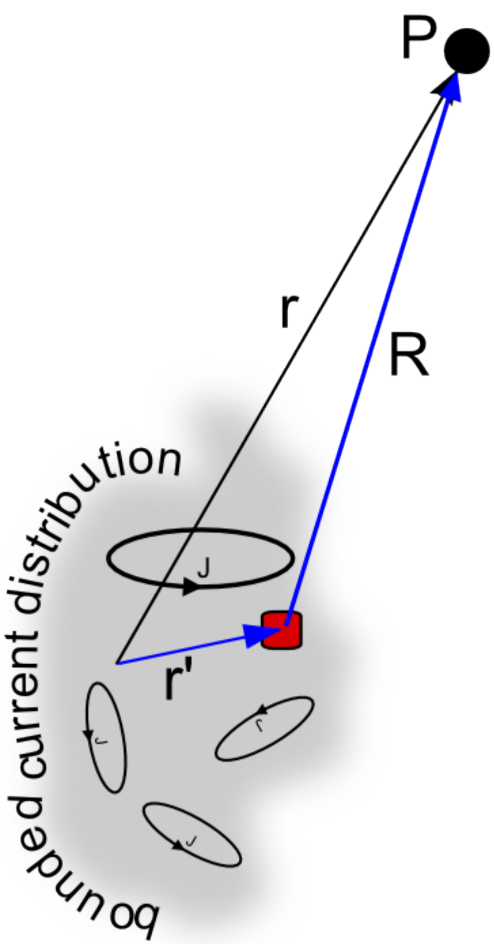
$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\frac{3\frac{\vec{r}}{|\vec{r}|}\left(\vec{m}\cdot\frac{\vec{r}}{|\vec{r}|}\right)-\vec{m}}{|\vec{r}|^3}=\boxed{\frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{|\vec{r}|^3}}\ *$$

- We should compare it with the expression for the field of **electric dipole** [7, 14]:

$$\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\frac{3\hat{r}(\vec{p}\cdot\hat{r})-\vec{p}}{|\vec{r}|^3}$$

*magnetic induction from first two terms of the expansion

- The values of the components of the successive terms of the multipole expansion of the field depend in general on the *origin of the coordinate system*.
- The dipole moment of the current distribution does not depend on the origin.
- It can be shown that quadrupole moments of the current distribution do not depend on origin provided that the dipole moment is zero.



$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\frac{3\frac{\vec{r}}{|\vec{r}|}\left(\vec{m}\cdot\frac{\vec{r}}{|\vec{r}|}\right)-\vec{m}}{|\vec{r}|^3}=\boxed{\frac{\mu_0}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{|\vec{r}|^3}}\,^*$$

- We should compare it with the expression for the field of **electric dipole** [7, 14]:

$$\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\frac{3\hat{r}(\vec{p}\cdot\hat{r})-\vec{p}}{|\vec{r}|^3}$$

*magnetic induction from first two terms of the expansion

Field of a distribution of magnetic moments*

- We are looking for the field produced by the distribution of spatially limited closed current loops. Remembering the definition of magnetization:

$$\vec{m}=\frac{1}{2}\int(\vec{r}\,'\times\vec{J})\,d^3r\,'=\int\vec{M}(\vec{r}\,')\,d^3r\,'$$

- We obtain potential at ***r*** from magnetic moments localized at ***r'***-s:

$$\vec{A}_{magn.dipol}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{M}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r\,' \quad \Longleftarrow \quad \vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\frac{1}{|\vec{r}|^3}\vec{m}\times\vec{r}$$

- Further, using first $\nabla'\left(\frac{1}{|r-r'|}\right)=\frac{\vec{r}-\vec{r}\,'}{|(r-r')|^3}$ and then $\nabla\times(f\vec{a})=f\nabla\times\vec{a}+\nabla f\times\vec{a}$ we rewrite the integrand:

$$\vec{M}(\vec{r}\,')\times\nabla'\left(\frac{1}{|r-r'|}\right)=\frac{1}{|r-r'|}\nabla'\times\vec{M}(\vec{r}\,')-\nabla'\times\frac{\vec{M}(\vec{r}\,')}{|r-r'|}$$

$$\vec{M}(\vec{r}\,')\times\nabla'\left(\frac{1}{|r-r'|}\right)=-\nabla'\left(\frac{1}{|r-r'|}\right)\times\vec{M}(\vec{r}\,')$$

to obtain:

$$\vec{A}_{magn.dipol}(\vec{r})=\frac{\mu_0}{4\pi}\int\left(\frac{1}{|r-r'|}\nabla'\times\vec{M}(\vec{r}\,')-\nabla'\times\frac{\vec{M}(\vec{r}\,')}{|r-r'|}\right)d^3r\,'$$

*this section is taken from K.J. Ebeling and J. MJ. Mähnß [14]

Field of a distribution of magnetic moments

- We assume that moments associated with current loops occupy a finite volume (i.e. **magnetization vanishes at infinity**) and integrate first the second term of the integral.

$$\int \nabla' \times \frac{\vec{M}(\vec{r}\,')}{|r-r'|} d^3 r' \Rightarrow \int_{-\infty}^{+\infty} \left[\frac{\partial}{\partial y'} \frac{\vec{M}_z(\vec{r}\,')}{|r-r'|} - \frac{\partial}{\partial z'} \frac{\vec{M}_y(\vec{r}\,')}{|r-r'|} \right] dx'\,dy'\,dz' =$$

x-component of the curl

$$\iint_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{\partial}{\partial y'} \frac{\vec{M}_z(\vec{r}\,')}{|r-r'|} dy' \right] dx'\,dz' - \iint_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{\partial}{\partial z'} \frac{\vec{M}_y(\vec{r}\,')}{|r-r'|} dz' \right] dx'\,dy' =$$

integral of the derivative

$$\iint_{-\infty}^{+\infty} \underbrace{\left[\frac{\vec{M}_z(\vec{r}\,')}{|r-r'|} \right]_{-\infty}^{+\infty}}_{0} dx'\,dz' - \iint_{-\infty}^{+\infty} \underbrace{\left[\frac{\vec{M}_y(\vec{r}\,')}{|r-r'|} \right]_{-\infty}^{+\infty}}_{0} dx'\,dy' = 0 \Rightarrow \int \nabla' \times \frac{\vec{M}(\vec{r}\,')}{|r-r'|} d^3 r' = 0$$

the second term of integral is zero

- Finally for a contribution of the magnetization to the magnetic potential ***A*** we have:

$$\boxed{\vec{A}_{magn.dipol}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3 r'}$$

*this section is taken from K.J. Ebeling and J. MJ. Mähnß [14]

Field of a distribution of magnetic moments

- The overall potential ***A*** can be written as**:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{j}_{free}(\vec{r}\,')+\nabla'\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'$$

The effect of magnetic moment distribution on magnetic field is the same as that of current distribution given by:

$$\vec{j}_{bound}(\vec{r})=\nabla\times\vec{M}(\vec{r})$$

We distinguish two types of currents contributing to magnetic field:

- the free currents – flowing in lossy circuits (coils, electromagnets) or superconducting coils; in general one can influence (switch on/off) and measure free currents
- the bound currents – due to intratomic or intramolecular currents and to magnetic moments of elementary particles with spin [13]

**A was defined previously (p. 18) as:

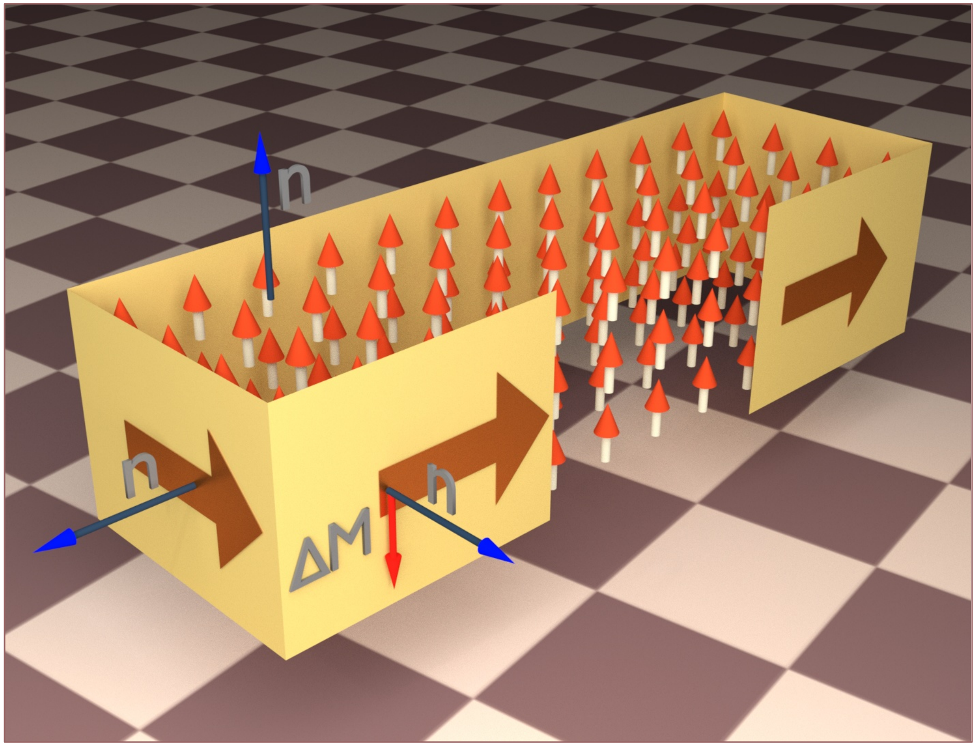
$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{j(\vec{r}\,')}{|r-r'|}d^3r'$$

 *this section is taken from K.J. Ebeling and J. MJ. Mähnß [14]

Field of a distribution of magnetic moments

- The overall potential ***A*** can be written as**:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{j}_{free}(\vec{r}\,')+\nabla'\times\vec{M}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d^3r'$$



The effect of magnetic moment distribution on magnetic field is the same as that of current distribution given by:

$$\vec{j}_{bound}(\vec{r})=\nabla\times\vec{M}(\vec{r})$$

If the magnetic moments density is discontinuous the equivalent (to the moments) surface current distribution is given by [9]:

$$\vec{K}_{bound}(\vec{r})=\vec{n}\times\Delta\vec{M}(\vec{r}),$$

where normal **n** is directed from region one to region two and **ΔM** is defined as:

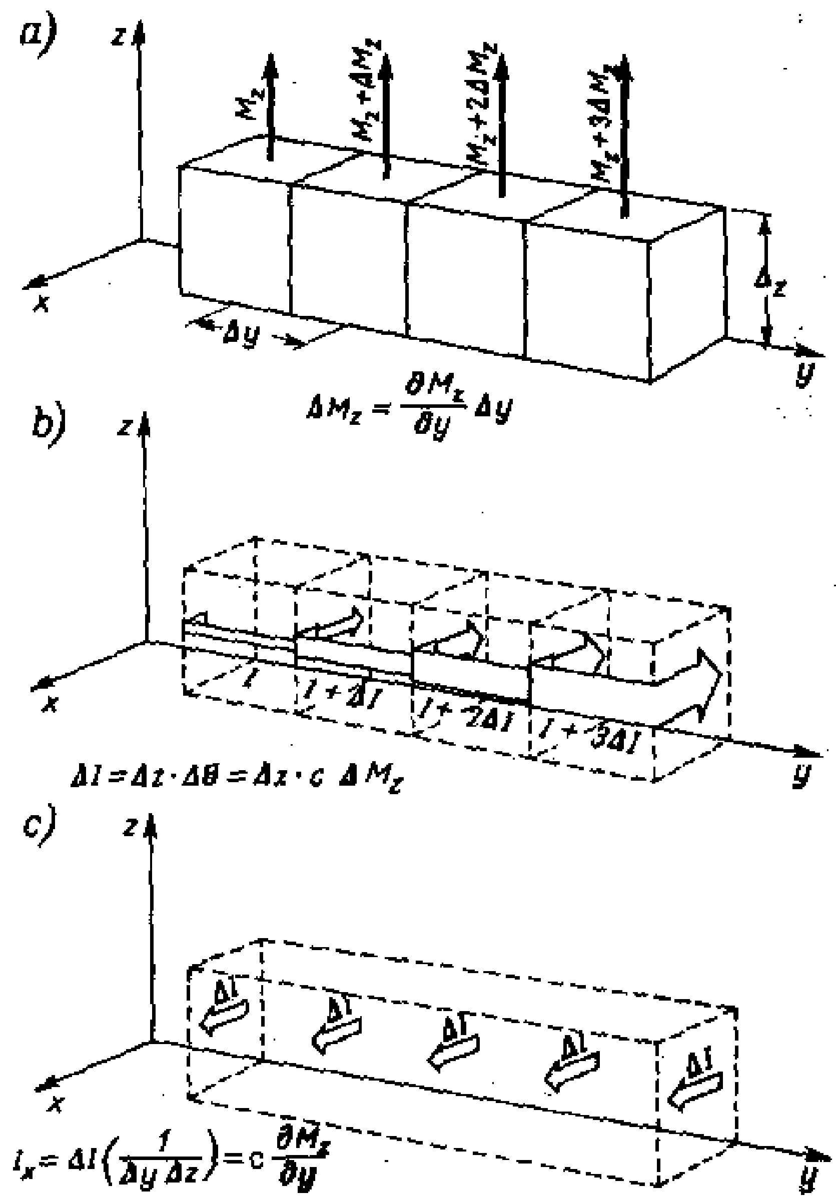
$$\Delta\vec{M}(\vec{r})=\vec{M}_2(\vec{r})-\vec{M}_1(\vec{r}).$$

Field of a distribution of magnetic moments

- An illustration of the fact that:

$$\vec{j}_{bound}(\vec{r}) = \nabla \times \vec{M}(\vec{r})$$

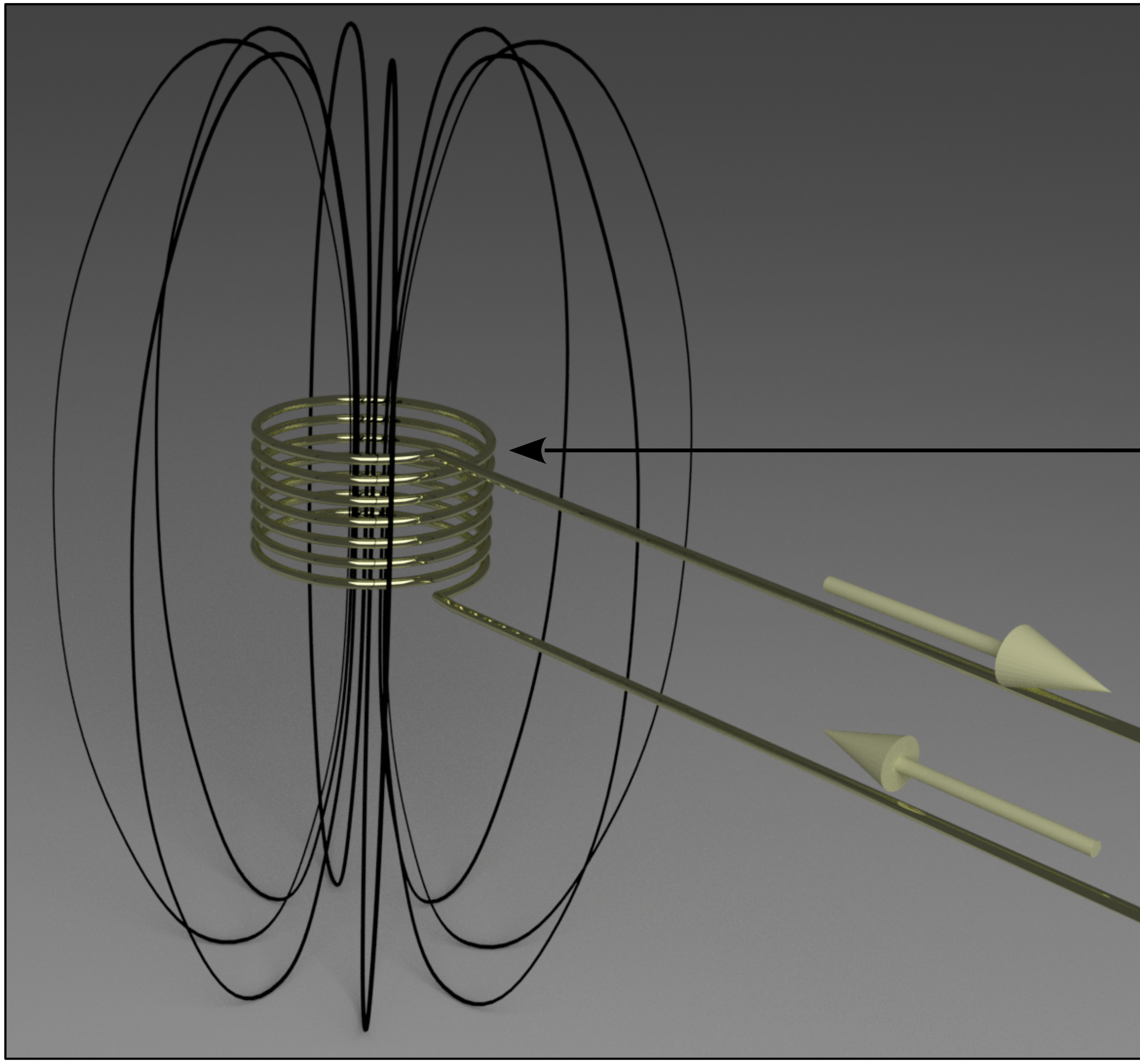
$$\nabla \times \vec{M}(\vec{r}) \neq 0 \rightarrow \vec{M}(\vec{r}) \neq const$$



Rys. 10.21. Namagnesowanie niejednorodne jest równoważne gęstości prądu objętościowego.

image source E. M. Purcell, Elektryczność i Magnetyzm, PWN Warszawa 1971

Correspondence between the coil and the magnet



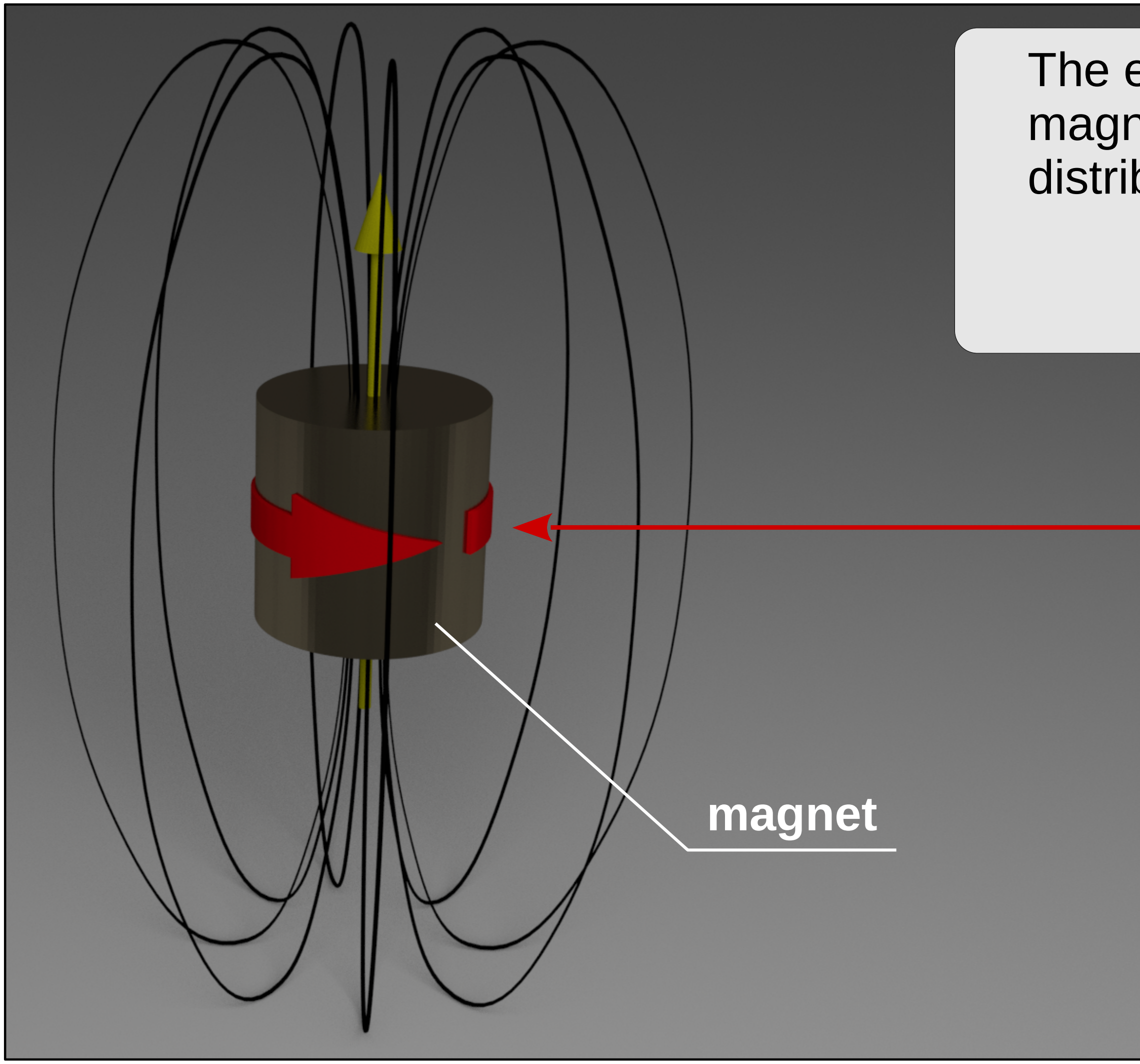
the typical coils has up to several hundred windings and the coil height is several tens of centimeters

the typical coils are fed with currents of up to several amperes

The surface current densities ***K*** in the coils are at most of the order of 100 thousand amperes per meter:

$$K = \text{several hundred} \times \text{several amperes} \times \frac{100\,\text{cm}}{\text{several tens of centimeters}} \approx 100\,\text{kA/m}$$

Correspondence between the coil and the magnet



The effect of magnetic moment distribution on magnetic field is the same as that of current distribution given by:

$$\vec{K}_{bound}(\vec{r})=\vec{n}\times\Delta\vec{M}(\vec{r}),$$

The surface current densities ***K*** corresponding to magnetization (magnets: alloys of iron, cobalt etc.) are of the order of **million ampere per meter**.

Introducing magnetic field strength H*

- From Biot-Savart law we have:

$$\nabla\times\vec{B}=\mu_0\vec{j}(\vec{r})=\mu_0\vec{j}_{free}+\mu_0\vec{j}_{bound}=\mu_0\vec{j}_{free}+\mu_0\nabla\times\vec{M} \qquad (6)$$

- We introduce a field strength vector:

$$\boxed{\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}}$$

In **old** cgs system:

$$\boxed{\vec{H}=\vec{B}-4\pi\vec{M}}$$

- From (6) we have:

$$\nabla\times\vec{B}-\mu_0\nabla\times\vec{M}=\mu_0\nabla\times(\frac{1}{\mu_0}\vec{B}-\vec{M})=\mu_0\vec{j}_{free}$$

- It follows that the rotation of field strength **H is determined solely by the free currents**:

$$\boxed{\nabla\times\vec{H}=\vec{j}_{free}}$$

- In general $\boxed{\nabla\cdot\vec{H}\neq 0}$ i.e. magnetic field strength is not source-free.

*this section is taken from K.J. Ebeling and J. MJ. Mähnß [14]

Magnetic moment of an electron

- Spin magnetic moment (Bohr magneton): $\mu_B = \dfrac{e h}{4\pi m_e} = 9.27400968(20)\times 10^{-24}\, A\, m^2$ *
- Magnetic moment of electron originates from spin i.e. angular momentum of electron which is equal to $\sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)}\dfrac{h}{2\pi}$ and its component along arbitrary direction can take on values $\pm\dfrac{1}{2}\dfrac{h}{2\pi}$.
- the magnitude of magnetic moment of electron is **constant**; only its orientation can be changed.
- **Giromagnetic ratio**: $\gamma = \dfrac{\vec{m}}{\vec{L}}$, $\vec{L}$ - angular momentum
- Giromagnetic ratio for a classical rigid body (with mass density proportional to charge density) equals $\gamma = \dfrac{q}{2m}$
- Giromagnetic ratio for spin magnetic moment is **twice** (**g_e- factor**) that of classic circular movement of a charge (like for examlpe electron circulating nucleus).

$$\gamma_e = 1.760859708(39)\times 10^{11}\, s^{-1}\, T^{-1}$$

*it is customary to express magnetic moment in J T^{-1} which is equivalent [1T=1 kg $s^{-2}A^{-1}$].

Magnetic moment of an electron

- Spin g_e factor:

$$\vec{m}_e = -g_e \frac{e}{2m} \vec{S} \qquad\qquad \vec{m} = \gamma \vec{L}$$

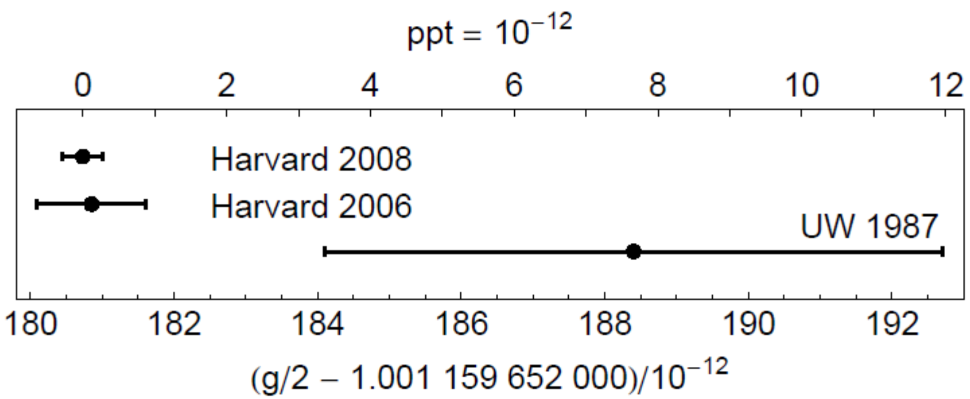


Fig. 6.1. Most accurate measurements of the electron $g/2$.

G. Gabrielse, Measurements of the Electron Magnetic Moment

- At large distances electron magnetic field has a dipolar character
- The external field exerts on electron the torque which is equal to the one exerted on the current loop with equal magnetic moment
- Within the electron $\nabla \vec{B} = 0$ as in classical sources of magnetic field [13].

The Rashba effect

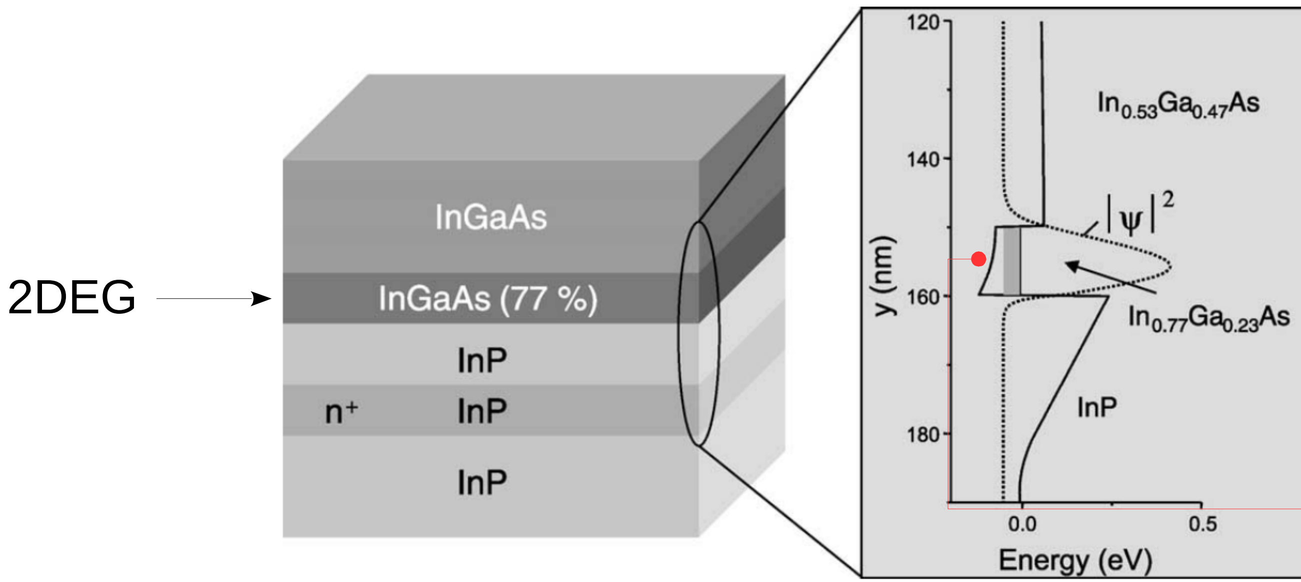


Fig. 2. Schematic illustration of the layer sequence of the InGaAs/InP heterostructure. The right panel shows the conduction band profile and the probability amplitude $|\Psi|^2$ of the envelope function.

image from Th. Schäpers, J. Knobbe, van der Hart, and H. Hardtdegen, Science and Technology of Advanced Materials **4**, 19 (2003)

- InP – semi-insulating, high carrier mobility
- InP lattice constant match $In_{0.53}Ga_{0.47}As$ at RT [15]
- $In_{0.77}Ga_{0.23}As$ layer is strained due to lattice mismatch

- the electron wave function is located mainly in the strained $In_{0.53}Ga_{0.47}As$ layer
- the electrons in the 2DEG layer come from negatively doped InP layer [11]
- the tilted potential profile results in an electric field in the quantum well [11]
- high mobilities in 2DEG ≈10^5 cm^2/Vs [(cm/s)/(V/m)] at 40K

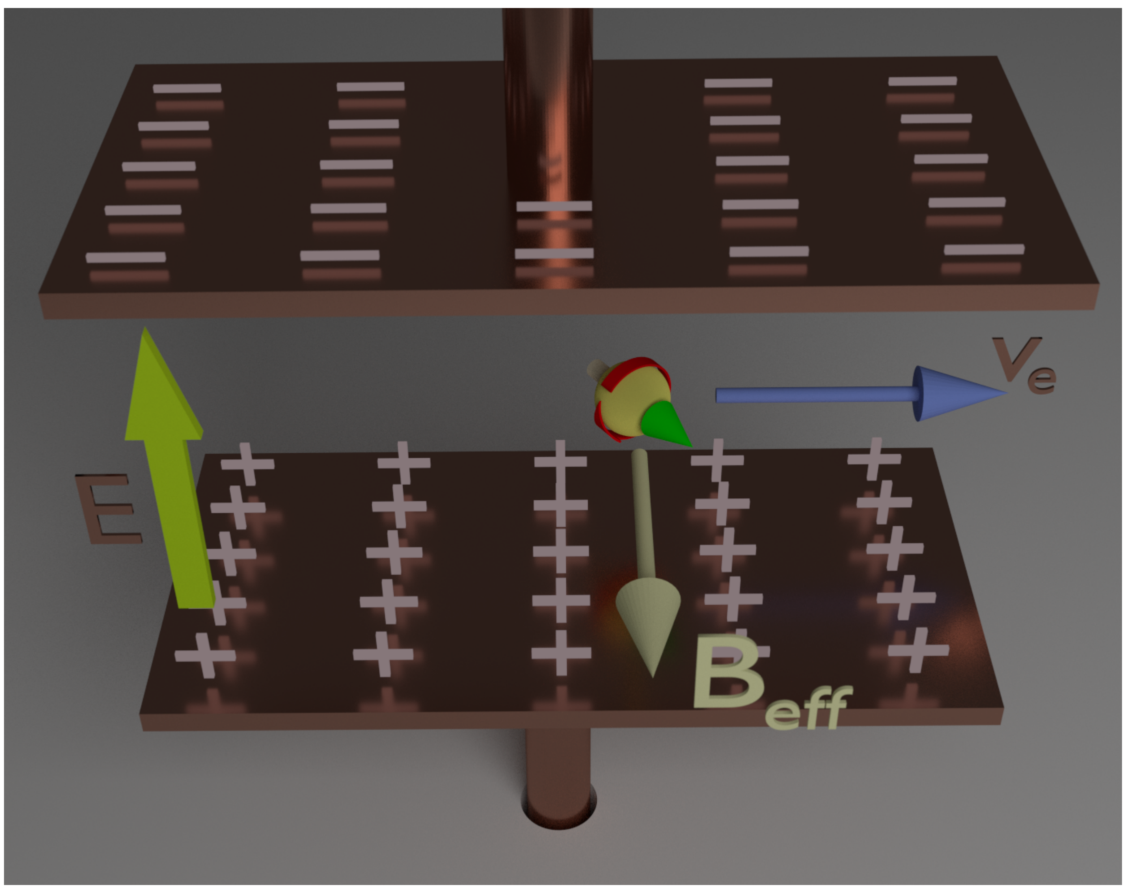
The Rashba effect

- the electric field in 2DEG is oriented perpendicularly to its plane
- electrons moving from the source to the drain experience the effective magnetic field given by Lorentz transformation [10]:

$$B'_{\parallel}=B \qquad B'_{\perp}=\frac{(\vec{B}-(\vec{v}/c^2))\times\vec{E}\perp}{\sqrt{(1-v^2/c^2)}} \rightarrow B'_{\perp}=\frac{(\vec{v}/c^2)\times\vec{E}}{\sqrt{(1-v^2/c^2)}}$$

we assume that there is no external magnetic field

- the magnetic field experienced by the electrons is oriented perpendicularly ($\vec{v}\times\vec{E}$) to the plane described by their velocity and the electric field



- charged plates are the source of a magnetic field experienced by moving electrons
- the electron spins precess in the magnetic field

The Rashba effect

- the electric field in 2DEG is oriented perpendicularly to its plane
- electrons moving from the source to the drain experience the effective magnetic field given by Lorentz transformation [10]:

$$B'_{\parallel}=B \qquad B'_{\perp}=\frac{(\vec{B}-(\vec{v}/c^2))\times\vec{E}\perp}{\sqrt{(1-v^2/c^2)}} \rightarrow B'_{\perp}=\frac{(\vec{v}/c^2)\times\vec{E}}{\sqrt{(1-v^2/c^2)}}$$

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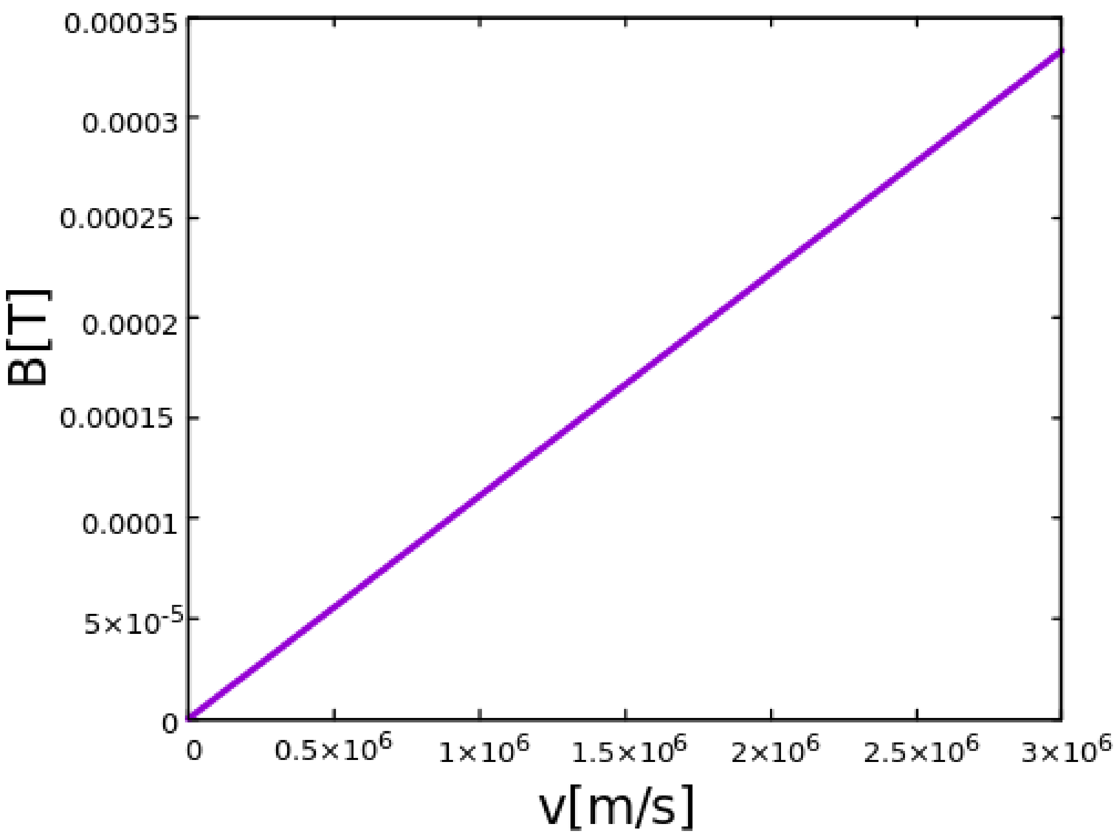


$E=10^7$ V/m

- in III-V semiconductor structures additional contribution to spin-orbit interaction comes from **Dresselhaus effect** caused by bulk inversion asymmetry [20]

- high mobilities in 2DEG ≈10^5 cm^2/Vs [(cm/s)/(V/cm)] (10 m^2/Vs) at 40K

Forces in magnetic field

- In some case the quantity of interest is not the field strength/induction but its spatial gradient:

- magnetophoresis
- magnetobiology (cell growth)

$$\chi=\frac{\vec{M}}{\vec{H}} \rightarrow \chi_p=\frac{M}{H} \qquad E=-\vec{m}\cdot\vec{B} \qquad \vec{m}=\vec{M}\cdot V \qquad V-\text{ volume of the magnet}$$

- the induced magnetic moment of a superparamagnet is parallel to the external field
- we bring the magnet from infinity (B=0) to the location with the magnetic field B and decrease thus its energy:

$$m=V\chi_p\frac{B}{\mu_0} \rightarrow dm=V\chi_p\frac{dB}{\mu_0} \rightarrow dE=-dm\,B=-V\chi_p\frac{dB}{\mu_0}B$$

$$E=-\int_0^{B(\vec{r})} V\chi_p\frac{B}{\mu_0}dB=-\frac{1}{2\mu_0}V\chi_p B^2$$

The force acting on a magnet is $\vec{F}=-\nabla E$:

$$\boxed{\vec{F}=\frac{1}{2\mu_0}V\chi_p\nabla B^2}$$

The higher the gradient of B^2 the higher the force acting on a para/superparamagnetic material

The formula is true within the field range in which susceptibility is constant

3D structures - a paternoster for superparamagnetic beads

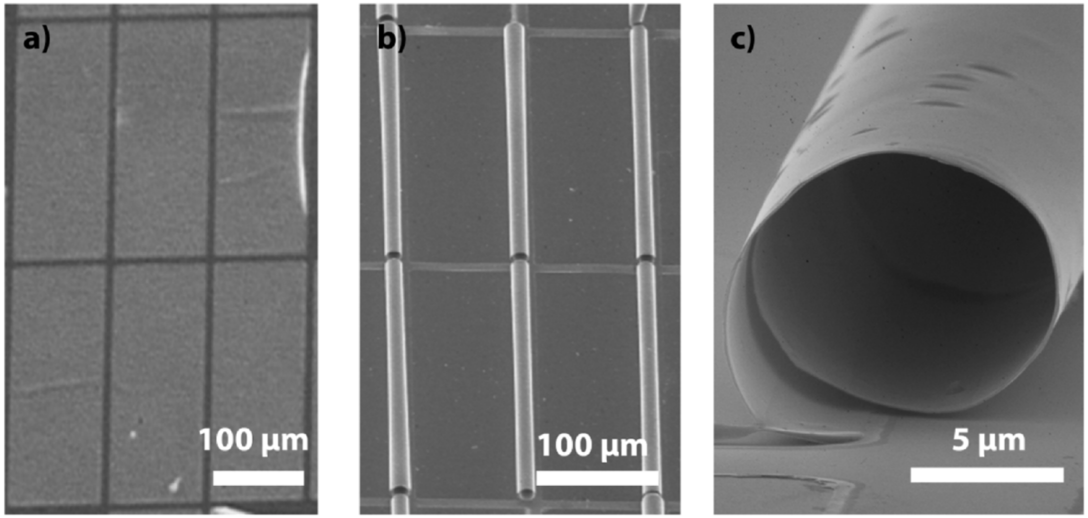
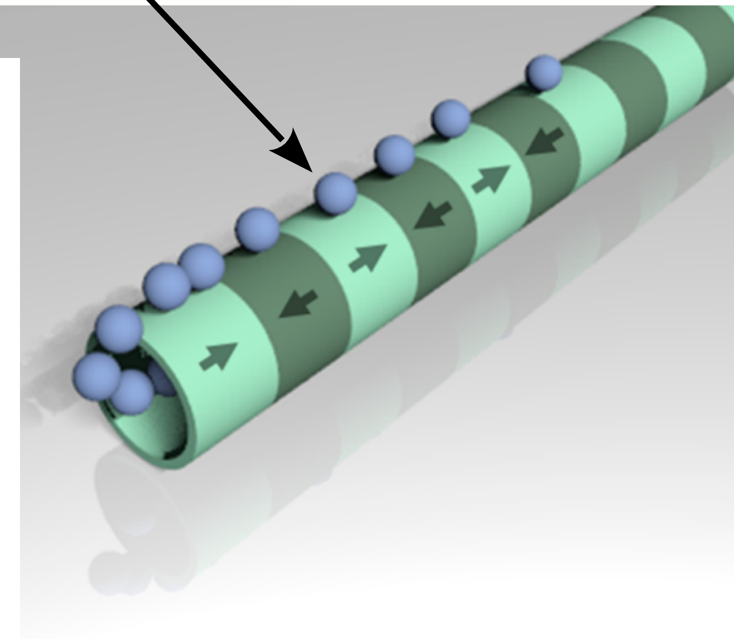


Figure 2. Scanning electron microscopy (SEM) images of 300 × 100 μm^2 sized prestrained layer systems (a) before and (b) after rolling up upon selective release from the substrate. (c) The magnetically stripe patterned exchange bias tubes possess a diameter of about 10 μm.



- exchange bias system: Cu(50nm)/$Ir_{17}Mn_{83}$(10nm)/$Co_{70}Fe_{30}$(7.5nm)/Ta (10nm) deposited via rf sputtering in an external magnetic field of 28 kA/m
- magnetic patterning (the direction of the exchange bias) done with He^+ ion bombardment of the films covered with patterned resist

images from T. Ueltzhöffer, R. Streubel, I. Koch, D. Holzinger, D. Makarov, O.G. Schmidt, and A. Ehresmann, ACS Nano **10**, 8491 (2016)

3D structures - a paternoster for superparamagnetic beads

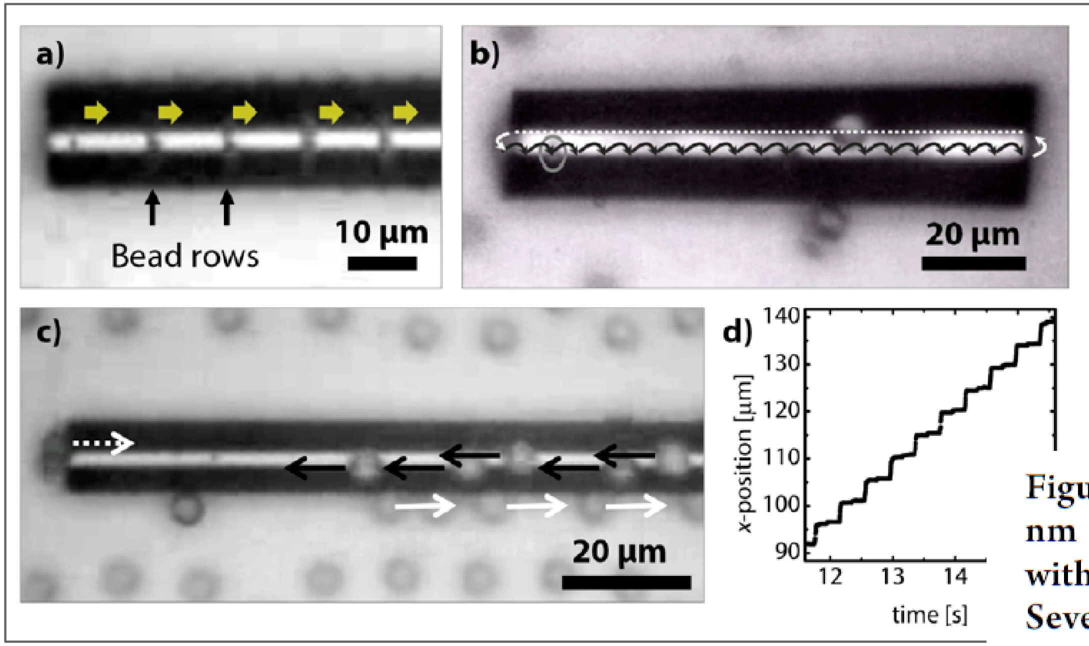


Figure 4. (a) Superparamagnetic beads with diameters of d_1 = 500 nm moving above the magnetically patterned exchange bias tube with a diameter of d_{Tube} = 10 μm (see Supplementary Video 1). Several beads are located close to each other and appear as lines occupying every second domain wall. With each magnetic field pulse of H_z = 5.5 mT, the relevant potential energy minima are shifted to the following domain wall, forcing the beads to move forward. (b) Agglomerate of two superparamagnetic beads each with a diameter of d_2 = 2 μm moving above and retracing inside the magnetically patterned exchange bias tube (see Supplementary Video 2). The black arrows indicate the 20 steps, each from one to another domain wall. The white dotted track indicates the way of retracing inside the tube without changing parameters. (c) Superparamagnetic beads with a diameter of d_3 = 6 μm moving above and next to the magnetically patterned exchange bias tube (see Supplementary Video 3). The direction of transport is reversed at the tube's entrance (white dotted arrow) and next to the tube (white arrows) compared to the one above the tube (black arrows). (d) Step profile of superparamagnetic beads with a diameter of d_2 = 2 μm (recorded from Supplementary Video 2), where the position is depicted versus the time.

image from T. Ueltzhöffer, R. Streubel, I. Koch, D. Holzinger, D. Makarov, O.G. Schmidt, and A. Ehresmann,
ACS Nano **10**, 8491 (2016)

Most important facts from todays talk:

- Static magnetic field sources are electric currents and intrinsic magnetic moments of elementary particles

- At distances large in comparison to its spatial extension every current distribution produces magnetic induction which can be approximated by magnetic dipole

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